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Pseudolocalized three-dimensional solitary waves as quasi-particles

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HIGHLIGHTS

- A higher-order dispersive equation is introduced as a candidate field theory.
- 3D wave profiles, which are only pseudolocalized, are found.
- Quasi-particles (QPs) are associated with the waves; the QPs' equations of motion are derived.
- The force of attraction goes as the inverse square of the distance between QPs at large separations.

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ABSTRACT

A higher-order dispersive equation is introduced as a candidate for the governing equation of a field theory. A new class of solutions of the three-dimensional field equation are considered, which are not localized functions in the sense of the integrability of the square of the profile over an infinite domain. For this new class of solutions, the gradient and/or the Hessian/Laplacian are square integrable. In the linear limiting case, an analytical expression for the pseudolocalized solution is found and the method of variational approximation is applied to find the dynamics of the centers of the quasi-particles (QPs) corresponding to these solutions. A discrete Lagrangian can be derived due to the localization of the gradient and the Laplacian of the profile. The equations of motion of the QPs are derived from the discrete Lagrangian. The pseudomass ("wave mass") of a QP is defined as well as the potential of interaction. The most important trait of the new QPs is that, at large distances, the force of attraction is proportional to the inverse square of the distance between the QPs. This can be considered analogous to the gravitational force in classical mechanics.

1. Introduction

The first half of the 20th century saw the formation of the current paradigm in physics, in which particles and fields are the main distinct forms of matter. Then naturally arose the question of the interconnection between these two facets of the physical reality. In the early 1960s, Skyrme [1] came up with the idea that a localized solution of a given field equation can be considered as a particle. Because of the formidable mathematical difficulties in multiple spatial dimensions, the new idea was demonstrated in 1D for the case of the *sine*–Gordon equation (*s*GE), for which an analytical solution was found in [2] for a profile composed by the superposition of two localized shapes. The shapes interacted (scattered) in a very similar fashion as two particles would do if they collided with each other. In seemingly unrelated research, Fermi and coworkers [3] discovered

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numerically that a virtually random initial condition for the difference equations modeling an atomic lattice tends, in the long term, to organize into a chain of localized shapes. Observing that the Korteweg–de Vries equation (KdVE) is the limit of the difference equations modeling the lattice, Zabusky and Kruskal [4] performed similar numerical experiments for KdVE and confirmed the tendency of the initially "thermalized" modes to organize into localized waves that interact as particles: the term *soliton* was introduced for this kind of wave.

Nowadays, the subject of solitons enjoys considerable attention. The surveys [5–7] give a good perspective of the wide scope of soliton research at the end of the 1970s. Applications to dynamics of atomic latices are summarized in [8]. In [9] an excellent updated survey of the application of the soliton concept in elasticity can be found. Several general surveys are also available [10–12]. The list of the equations for which soliton solutions are sought is ever expanding (see, e.g., [13]).

If an initial condition is composed of the superposition of two localized waves, the evolution of this wave system results in the virtual recovery of the shapes of the two initial localized waves after their collision but in shifted relative positions after leaving the site of interaction (see, e.g., [12,14,10,15]). The collision property is what justifies calling these localized waves "quasi-particles" (QPs). A QP solution of a fully integrable system is a soliton in the strict sense. For the case of a non-fully-integrable system, the term QP is safer, but many researchers use the term "soliton" in a broader sense that includes also non-fully-integrable cases. When the physical system is described by equation(s) that conserve the energy and momentum, then the same conservation properties are inherited by the QPs, as would be necessary for actual subatomic particles.

What would be called a "two-soliton" solution in the late 1960s, was given in 1962 by Perring and Skyrme [2] for the *s*GE, and the resemblance between the two-soliton solution and a specially constructed superposition of one-solitons could be seen in the cited paper. In [1], the localized solutions were qualitatively related to mesons assuming that the meson field is governed by the *s*GE. What is more important is that Skyrme introduced the fundamental notion that the localized (in an appropriate sense) solutions can be considered as particles of the field described by the particular equation under consideration. Using the techniques for finding analytically the two-soliton solution, an important analytical result was obtained in [16,17] by extracting the positions of the centers of the individual solitons from the actual two-soliton profile. Thus the trajectories of Skyrme's "particle" (the QP in the modern terminology) were found and shown to bend during the interaction. The last result completes, in a sense, Skyrme's proposal that the dynamics of the localized solution of the respective field equation can describe subatomic particle interactions. The impressive analytical success here can be clearly attributed to the full integrability of *s*GE and to the 1D nature of the solution. This line of research is currently actively pursued with the goal of creating a kind of "multi-body" formalism based on the connection to the field equations [18]. Thus, Skyrme's idea to establish the relation between the dynamics of soliton solutions of a field equation and subatomic particles proved to be a fruitful paradigm (see, e.g., [19]).

A similar approach has been developed for the integrable Camassa–Holm equation [20], making it possible to reduce the interaction dynamics of the so-called "peakons" to the solution of a system of Hamiltonian ordinary differential equations (ODEs) (see also [21] for an extension to cross-coupled Camassa–Holm equations). Yet another approach to the reduction of the dynamics of interacting solitons to a system of ODEs governing just a few parameters is available for the integrable nonlinear Schrödinger (NLS) equation, namely the adiabatic perturbation approach [22].

When the field equation is not fully integrable, the way to obtain information about the QP behavior of the solution is to make an ansatz consisting of the linear superposition of two one-soliton solutions with a priori unknown trajectories, assuming that the two-soliton solution can be fairly well approximated by the latter. Then, one derives a "discrete" Lagrangian with the trajectories as "generalized coordinates" and solves the Newtonian-like governing equations for the point dynamics of the centers of the individual solitons. This was proposed in [23,24] and applied to the (non-integrable) nonlinear Klein–Gordon equation (KGE). The trajectory function for the center of a QP was called a "collective coordinate" in [24] (see, also, [25]) and "collective variable" (CV) in [26]. Nowadays, this approach is known as *variational approximation* (VA). The success of the simplest approach with one CV inspired the introduction of more sophisticated VA models, some involving two (or more) CVs. An important step in this direction was made in [27], where the second CV is the "width" of the QP. Such a choice allows one to recover, as in [27,28], the relativistic dynamics of a single QP in the case of KGE (see [29] for multiple CVs). Unfortunately, the generalization of the 2-CV method to 2-QP interaction is not trivial. For this reason the 2-CV approach was tested in [26] for the *s*GE case. The integrals for the potential of interaction are unsurmountable without some simplifying assumptions (see [30]).

Despite this shortcoming, the VA is clearly a very promising way of establishing the wave–particle dualism for QPs, provided that some more pertinent models for the field equation are used. We would like to mention here that VA is one of the specific techniques to find an approximate solution of the field equations. The approach of reducing the system with distributed parameters to a discrete system amounts, in general, to a *coarse-grain description* (CGD) [31]. From this point of view the VA is the most physically consistent way to create a CGD of a continuous system because it retains the main conservation laws related to the latter.

Before proceeding further, it is important to mention that recently, another avenue for further development of the VA has been opened. The full two-soliton solution shows a significant deformation of the wave profile when the two main solitons are close to each other. Such a deformation is clearly not contained in the mere superposition of two one-soliton solutions. There is some utility to try to represent this deformation via the evolution of the widths of the QPs (as in the previous paragraph), but there is also another very practical approach: to consider what is left from the two-soliton solution when the one-soliton solutions are subtracted as a new particle that is briefly "born" at the site of interaction and promptly "dies" after the separation of the main lumps (the so-called "ghost particle" [32]). This idea was elaborated in [33] and offers a very

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