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On the role of nonlinearities in the Boussinesq-type wave equations

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HIGHLIGHTS

- The soliton-type solution found for the improved Heimburg–Jackson equation.
- The role of nonlinearities explained.
- The width of a soliton is governed by dispersive terms.

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ABSTRACT

Longitudinal mechanical waves in biomembranes are described by a Boussinesq-type wave equation. It is shown that in this case the nonlinearities are of a different type compared with conventional models of solids. The governing equation analysed in this paper is the improved Heimburg–Jackson model with two dispersive terms. The soliton-type solutions of such a wave equation are found and analysed. The existence of solitons depends on the ratio of nonlinear terms and the width of solitons is governed by dispersive terms.

1. Introduction

The Boussinesq approximation for water waves is known from the 19th century (see Bois [1]) and is nowadays generalised also for waves in solids [2,3]. In general terms, a Boussinesq-type model for modelling waves is based on the classical second-order wave operator but more effects are included. Altogether, it is characterised by the following effects [2]: (i) bi-directionality of waves (due to the second-order wave operator); (ii) nonlinearity (of any order); (iii) dispersion (of any order modelled by space and/or time derivatives of the fourth order at least). Such a model may be described by the following equation [2]:

$$u_{tt} - c_0^2 u_{xx} - \left(\frac{dF(u)}{du} \right)_{xx} = (\beta_1 u_{tt} - \beta_2 u_{xx})_{xx}, \tag{1}$$

where $F(u)$ is a polynomial, starting with second degree, c_0 is the velocity and β_1, β_2 are coefficients characterising dispersion. The crucial point for grasping the physical effects is certainly the structure of the nonlinear term in Eq. (1) and the signs of coefficients β_1, β_2 or the combination of possible other higher-order terms.

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Many studies are devoted to the dispersion analysis of Eq. (1), i.e., the influence of the structure of its r.h.s. [2,4–6] on dispersion relations and possible stabilities or instabilities of speeds over the large range of frequencies. Less attention is paid to the influence of nonlinearities on wave motion modelled by Eq. (1) or alike. Here the main issue is whether the nonlinear terms of $f(u)$-type or $g(u_x)$-type appear in the governing equation. Further we represent the governing equations either in terms of u (which is displacement) or in terms of $v = u_x$ (which is deformation). As shown further in Section 2, both cases have been described for modelling waves in various studies. However, the problem is not related to the formulation of governing equations only but also to the formulation of initial or boundary conditions, i.e., to the excitation of wave processes.

In this paper we start in Section 2 with the presentation of various models and then proceed to one of the crucial problems in solitonics — the existence of solitons (Section 3). Then in Section 4 we proceed to the analysis of the improved Heimburg–Jackson model and demonstrate how the solution depends on the shape of the 'pseudo-potential'. This explains the role of the $f(u)$-type nonlinearities in the model. It is also shown how the width of the soliton depends on dispersive effects. Finally, in Section 5, the final remarks are given.

2. Boussinesq-type models

In what follows, the main cases of Boussinesq-type models used for describing waves in solids are presented. For microstructured solids the governing equation for longitudinal waves in terms of displacement u is [3,7]

$$u_{tt} - (b + \mu u_x)u_{xx} = \delta(\beta u_{tt} - \gamma u_{xx})_{xx}, \tag{2}$$

where nonlinearity of the macrostructure is taken into account, b, μ, β, γ are physical coefficients and δ is a scale factor. In terms of deformation $v = u_x$, Eq. (2) takes the form

$$v_{tt} - bv_{xx} - \frac{1}{2}\mu\left(v^2\right)_{xx} = \delta(\beta v_{tt} - \gamma v_{xx})_{xx}. \tag{3}$$

The analysis of (2) and (3) is given by Tamm [8], Peets [9] and Berezovski et al. [10]. For longitudinal waves in rods, the governing equation in terms of deformation is derived by Porubov [11]. In original notations this equation is

$$v_{tt} - av_{xx} - c_1\left(v^2\right)_{xx} = -\alpha_3 v_{xxtt} + \alpha_4 v_{xxxx} \tag{4}$$

with a, c_1, α_3, α_4 denoting the physical coefficients including the radius of the rod. Here $v = u_x$. Compared with Eq. (3), where dispersion effects appear due to the presence of the microstructure, the dispersion effects in Eq. (4) are due to the geometry of the rod.

In mathematical terms, dispersive effects in Eqs. (1)–(3) are described by the higher-order space and space time derivatives. The structure of Eq. (2) demonstrates clearly the influence of the inertia of the microstructure and its elasticity [3] while in the case of geometrical dispersion the mixed derivative appears due to the Love assumption linking the transverse displacement w to the longitudinal deformation u_x: $w = -r\nu u_x$, where r is the radius of the rod and ν – Poisson's ratio [11].

It is possible that the dispersive effects are described by different assumptions. Bogdanov and Zakharov [6] have used the following form (in original notations)

$$\frac{3}{4}\alpha^2 v_{tt} - \beta v_{xx} + \frac{3}{2}\left(v^2\right)_{xx} = -\frac{1}{4}v_{xxxx} \tag{5}$$

in order to study long-wave and short-wave instabilities. Here α and β are the physical coefficients.

Christou and Papanicolaou [12] have studied even higher-order dispersion modelled by

$$v_{tt} - \gamma^2 v_{xx} - \alpha_1\left(v^2\right)_{xx} = \beta_1 v_{4x} + \delta_1 v_{6x}, \tag{6}$$

where $\gamma, \alpha_1, \beta_1, \delta_1$ are coefficients with $\delta_1 > 0$. Maugin [13] has introduced the Maxwell–Rayleigh equation

$$u_{tt} - u_{xx} - \left(\frac{dF(u)}{du}\right)_{xx} = \gamma(u_{xx} - u_{tt})_{tt}, \tag{7}$$

where dispersive effects are influenced by the fourth-order time derivatives.

The nonlinearities may be modelled also differently. For waves in biomembranes, Heimburg and Jackson [14] have assumed that the sound velocity in the membrane depends on the density changes $\Delta\rho^A = u$:

$$c^2 = c_0^2 + pu + qu^2 + \cdots, \tag{8}$$

where c_0 is the velocity in the unperturbed state, $p < 0$ and $q > 0$ are coefficients determined from the experiments [14]. Then the longitudinal waves are described by the governing equation

$$u_{tt} - \left[\left(c_0^2 + pu + qu^2\right)u_x\right]_x = -h_1 u_{xxxx}, \tag{9}$$

where h_1 characterises the strength of dispersion described by an added *ad hoc* term (u_{xxxx}).

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