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New families of pure gravity waves in water of infinite depth

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HIGHLIGHTS

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- Nonlinear Waves.
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ABSTRACT

Nonlinear periodic gravity waves propagating at a constant velocity at the surface of a fluid of infinite depth are considered. The fluid is assumed to be inviscid and incompressible and the flow to be irrotational. It is known that there are both regular waves (for which all the crests are at the same height) and irregular waves (for which not all the crests are at the same height). We show numerically the existence of new branches of irregular waves which bifurcate from the branch of regular waves. Our results suggest there are an infinite number of such branches. In addition we found additional new branches of irregular waves which bifurcate from the previously calculated branches of irregular waves.

1. Introduction

Nonlinear periodic gravity water waves propagating at the surface of a fluid at a constant velocity c have been studied for over 150 years and many interesting theoretical results have been obtained (see [1] for a review and references). Such waves can be studied by choosing a frame of reference moving with the wave speed c and by assuming that the flow is potential and steady. For simplicity only waves in water of infinite depth are considered in this paper.

Stokes [2,3] constructed a solution in the form of an expansion in powers of the amplitude of the wave and calculated the first few terms. Based on his results, he conjectured that, as the amplitude of the wave increases, the family of solutions will ultimately reach a limiting configuration with a stagnation point at the crests of the waves with an enclosed angle of 120°. This conjecture was confirmed numerically by Michell [4]. Early theoretical work on the subject can be found in Levi-Civita [5] and Struik [6]. A rigorous proof of the existence of the limiting configuration was provided in [7].

Detailed numerical studies of the properties of gravity waves became possible in the second half of the 20th century. In particular the computations in [8–11] show that for very steep waves (i.e. waves close to the limiting configuration), c is not a monotonic function of the steepness

$$s = \frac{H}{\lambda}. \tag{1}$$

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Here H is the difference of heights between a crest and a trough and λ is the wavelength. More precisely it was found that c goes through an infinite succession of maxima and minima as the limiting configuration is approached. These waves were the only type of waves known up to 1980. We shall refer to them as the branch of regular waves.

Chen and Saffman [12] (see also Saffman [13]) discovered numerically new branches of solutions which bifurcate from the branch of regular waves. They refer to them as irregular waves of type n. They are characterised by n crests per wavelength with $n-1$ crests at a different height than the remaining one. They calculated irregular waves of type 2 and 3 explicitly. These waves bifurcate from a train of regular waves of finite amplitude whose wavelengths is n times the fundamental wavelength. They have limiting configurations for which the highest crests are stagnation points with an enclosed angle of 120°.

Olfe and Rottman [14] calculated directly the limiting configurations of irregular waves by generalising the method used by Michell [4] for the limiting configuration of regular waves. They obtained results for irregular waves of type 2, 3 and 4.

Vanden-Broeck [15] extended the results in [12,14,15] to larger value of n (up to $n = 9$). He provided numerical evidence that the irregular waves approach a 'generalised solitary wave' as $n \to \infty$. Here a generalised solitary wave refers to a non-periodic wave characterised by a train of ripples in the far field.

In this paper we study further the properties of irregular waves. For simplicity we limit our attention to the case $n = 2$. We provide numerical evidence that in addition to the bifurcation point discovered by Chen and Saffman [12], there are an infinite number of similar bifurcation points. Each bifurcation point occurs between two successive extrema (maximum or minimum) of the branch of regular wave. In addition we follow numerically the first branch of irregular wave with $n = 2$ (i.e. the one computed by Chen and Saffman [12]) up to the limiting configuration with a 120° angle at the crests and we show that this branch oscillates also infinitely often as the limiting configuration is approached. This suggests that there might be further bifurcation points between each pair of successive extrema. This is confirmed by our numerical calculations.

2. Formulation

We consider a train of two-dimensional periodic waves travelling at a constant velocity c at the surface of a fluid of infinite depth. We take a frame of reference moving with the wave, so that the flow is steady. At infinite depth the flow is characterised by a uniform flow with a constant velocity c. We choose cartesian coordinates with the y-axis directed vertically upwards and $x = 0$ at a crest or a trough of the wave. Gravity is acting in the negative y-direction.

We assume that the fluid is incompressible and inviscid and that the flow is irrotational. Therefore we can introduce a potential function $\phi(x, y)$ and a streamfunction $\psi(x, y)$. Without loss of generality we choose $\psi = 0$ on the free surface and $\phi = 0$ at $x = 0$. We also define the complex potential

$$f = \phi + i\psi \tag{2}$$

and the complex function

$$\delta + i\beta = \frac{1}{u - iv}. \tag{3}$$

Here u and v are the horizontal and vertical components of the velocity.

On the free surface the pressure is constant and the dynamic boundary condition yields

$$u^2 + v^2 + 2gy = B. \tag{4}$$

Here B is the Bernoulli constant. We choose the origin of y to correspond to the level of the free surface where the velocity is equal to c. This forces $B = c^2$.

We use ϕ and ψ as independent variables. It then follows that $\delta + i\beta$ is an analytic function of f in the lower half plane $\psi < 0$ of the complex f-plane. (see for example Vanden-Broeck [1]). Furthermore

$$x_\phi + iy_\phi = \delta + i\beta \tag{5}$$

where the subscript ϕ denotes the partial derivative with respect to ϕ.

We now introduce dimensionless variables by using the wavelength λ as the unit length and c as the unit velocity. The dynamic boundary condition (4) becomes in dimensionless variables

$$\frac{1}{[\tilde{\delta}(\phi)]^2 + [\tilde{\beta}(\phi)]^2} + y_0 + \frac{4\pi}{\mu}\int_0^\phi \tilde{\beta}(\varphi)d\varphi = 1 \tag{6}$$

where y_0 is the value of y at $\psi = \phi = 0$ and

$$\mu = \frac{2\pi c^2}{g\lambda} \tag{7}$$

is ta dimensionless squared Froude number. Here $\tilde{\delta}(\phi)$ and $\tilde{\beta}(\phi)$ denote the values of δ and β on the free surface $\psi = 0$. We have used (5) in (6) to express y in terms of $\tilde{\beta}(\phi)$.