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Diffusion of elastic waves in a two dimensional continuum with a random distribution of screw dislocations

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HIGHLIGHTS

- Diffusion of elastic waves by screw dislocations is studied in two dimensions.
- Incoherent behavior is characterized by way of a Bethe–Salpeter equation.
- A Ward–Takahashi identity is demonstrated.
- The diffusion coefficient is calculated.

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ABSTRACT

We study the diffusion of anti-plane elastic waves in a two dimensional continuum by many, randomly placed, screw dislocations. Building on a previously developed theory for coherent propagation of such waves, the incoherent behavior is characterized by way of a Bethe–Salpeter (BS) equation. A Ward–Takahashi identity (WTI) is demonstrated and the BS equation is solved, as an eigenvalue problem, for long wavelengths and low frequencies. A diffusion equation results and the diffusion coefficient D is calculated. The result has the expected form $D = v^* l/2$, where l, the mean free path, is equal to the attenuation length of the coherent waves propagating in the medium and the transport velocity is given by $v^* = c_T^2/v$, where c_T is the wave speed in the absence of obstacles and v is the speed of coherent wave propagation in the presence of dislocations.

1. Introduction

The interaction of elastic waves with dislocations in elastic solids has been studied over several decades, since the pioneering work of Nabarro [1], Eshelby [2,3], Granato and Lücke [4,5] and Mura [6]. Nabarro [1] and Eshelby [2,3] noted that the mathematics describing the interaction of screw dislocations with anti-plane elastic waves in a two-dimensional continuum is the same as that of electromagnetic waves in interaction with point charges, also in two dimensions. A translation of the well-studied electrodynamics allowed these researchers to obtain expressions for the elastic radiation emitted by a screw dislocation in arbitrary motion. The more difficult case of the radiation generated by an edge dislocation in arbitrary motion was solved by Mura [6], who also found expression for the radiation generated by a dislocation loop of arbitrary shape undergoing also arbitrary motion. The converse problem, the response of a dislocation loop to an incoming, time dependent stress wave, was solved by Lund [7] using arguments of energy and momentum conservation.

Granato and Lücke [4,5] formulated a theory for the propagation of an averaged acoustic wave in the presence of many dislocation segments, using a string model for the dislocation pioneered by Kohler [8]. Their results have been widely used

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to interpret experimental results of mechanical damping and modulus change that are both frequency and strain-amplitude dependent. This theory is a scalar theory that cannot distinguish between longitudinal and transverse waves, nor between edge and screw dislocations. Hence, it could not account for results that depend on the polarization of the waves. Also, attempts at using it to explain thermal conductivity measurements did not take into account the attenuation due to loss of coherence nor the diffusive behavior of incoherent contributions [9,10].

In recent years, Maurel, Lund, and collaborators have revisited the theory of the dislocation–wave interaction. Using the equations of [7] they obtained explicit expressions for the scattering of an elastic wave by screw and edge dislocations in two dimensions [11], and by pinned dislocation segments [12] and circular dislocation loops [13] in three. In addition, and using multiple scattering methods that go back to the work of Foldy [14], Karal and Keller [15] and Weaver [16], they obtained expressions for the effective velocity and attenuation of an elastic wave moving through a maze of randomly placed dislocations in two [17] and three [18] dimensions. These results generalized the theory of Granato–Lücke to keep track of the wave polarization and vector character of the dislocations. In this way previously unexplained experimental results concerning the different response of materials to longitudinal and transverse loadings found an explanation [19]. More interestingly, they suggested a way to use ultrasound as a non-intrusive wave to measure dislocation density. This possibility has been recently shown to be feasible, and leads to measurements more accurate than those obtained with X-ray diffraction [20,21]. The multiple scattering theory of elastic waves has also been studied in relation to propagation in polycrystals [22–24] and composites [25,26].

Having studied the behavior of coherent waves, the question naturally arises as to the behavior of incoherent waves. Diffusion techniques developed to deal with the localization of de Broglie waves that describe the quantum mechanics of electrons in interaction with randomly placed scatterers have been used to study the behavior of classical waves, a central role being played by the Bethe–Salpeter (BS) equation [27–29]. The behavior of elastic waves in interaction with a variety of scatterers and within a variety of geometries has been studied by Kirkpatrick [30], Weaver [16], and Van Tiggelen and collaborators [31–33]. Also, an early approach that uses energy transport equations [34] has been widely applied.

The diffusion of elastic waves, albeit in their quantized form, phonons, is central to thermal transport in materials, a topic of much current concern. Yet, surprisingly little appears to be quantitatively known about the role played by the interaction of elastic waves with dislocations [35] in thermal transport. It stands to reason then that a detailed study of the diffusive behavior of elastic waves in interaction with many, randomly placed, dislocations, should be undertaken. Since elastic waves are vector waves and dislocations are linear objects characterized by their tangent and Burgers vectors, the tensor algebra associated with the proposed study appears daunting. A first step should be to consider a simplified setting that captures the essence of the problem. This paper is devoted to precisely this aim: it studies the diffusive behavior, in two dimensions, of an anti-plane elastic wave in interaction with many, randomly placed, screw dislocations.

1.1. *Organization of this paper*

This paper is organized as follows: Section 2 recalls existing results for the interaction of anti-plane waves with screw dislocations. It includes the behavior of coherent waves when many such dislocations are present, as well as a recent result of Churochkin et al. [36] concerning the summability to all orders of the perturbation expansion needed to make sense of the theory. Section 3 constructs the Bethe–Salpeter equation and establishes the Ward–Takahashi identity that relates the coherent with the incoherent kernels. Section 4 solves the BS equation in the low frequency limit needed to obtain a diffusion behavior and an expression for the diffusion coefficient (Eq. (97)) is obtained. Section 5 has a discussion and final comments. A number of computations are carried out in several Appendices.

2. Interaction of anti-plane elastic waves with screw dislocations

The interaction of an anti-plane wave with a single dislocation was studied by Eshelby [2], Nabarro [1] and by Maurel et al. [11]. The coherent behavior that emerges when an anti-plane wave interacts with many, randomly located screws, was elucidated by Maurel et al. [17], using the following equation of motion in the frequency domain:

$$\left(\nabla^2 + k_\beta^2\right) v(\vec{x}, \omega) = -V_{(0)}(\vec{x}, \omega) v(\vec{x}, \omega) \tag{1}$$

where v is particle velocity as a function of two-dimensional position $\vec{x}$ and frequency ω, $k_\beta^2 = \omega^2/c_T^2$ with $c_T^2 = \mu/\rho$, and the potential $V_{(0)}$ is

$$V_{(0)}(\mathbf{x}, \omega) = \sum_{n=1}^{N} \mathcal{A}_{(0)}^n \, \frac{\partial}{\partial x_a} \delta(\vec{x} - \vec{X}_0^n) \, \frac{\partial}{\partial x_a} \bigg|_{\vec{x}=\vec{X}_0^n}, \tag{2}$$

and

$$\mathcal{A}_{(0)}^n = \frac{\mu}{M} \left(\frac{b^n}{\omega}\right)^2. \tag{3}$$