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On possible and necessary inclusion of intuitionistic fuzzy sets

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ABSTRACT

The problem of measuring the degree of inclusion for two intuitionistic fuzzy sets (IF-sets) is considered. Since the nature of IF-sets includes uncertainty expressed both by the membership and nonmembership functions, this very feature should be also regarded in the construction of the subsethood measures. Therefore, we suggest not a single coefficient measuring the grade of subsethood but two indicators that leave some space to the mentioned uncertainty and correspond rather to necessary and possible inclusion.

1. Introduction

Let A and B denote two sets in a universe of discourse $\mathbb{X}$. In classical set theory we say that a set A is a subset of B and we write $A \subset B$ if every element of A is an element of B, i.e.

$$A \subset B \iff (x \in A \Rightarrow x \in B) \quad \forall x \in \mathbb{X}. \tag{1}$$

Assuming that χ_A and χ_B are characteristic functions of sets A and B, respectively, condition (1) is equivalent to

$$A \subset B \iff \chi_A(x) \leqslant \chi_B(x) \quad \forall x \in \mathbb{X}. \tag{2}$$

For any set B we get $\varnothing \subset B$. Two sets are equal if and only if $A \subset B$ and $B \subset A$. Moreover, if $A \subset B$ and $\varnothing \neq A \neq B$ then A is said to be a proper subset of A.

However, starting from the seminal paper by Zadeh in 1965 introducing fuzzy sets, the problem of set inclusion understanding appeared again. It seems that in fuzzy environment instead of binary statement: being or not being a subset, it would be more natural to say, e.g. that A is "more or less" a subset of B and to indicate the degree to which B contains A. Thus various researches suggested different inclusion indicators, also called subsethood measures (see, e.g., [3–5,12,16,23]) and discussed axioms that such measure should fulfill (see [8,19,24]).

Since in a real life language negation not always identifies with the logical negation, Atanassov introduced intuitionistic fuzzy sets (IF-sets) in 1980. These sets are characterized by two functions: membership and nonmembership function which are not necessarily complementary. Thus IF-sets seem to be very useful for modelling situations with missing information or

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hesitance, so typical, e.g., in decision making. But because of the rapid growth of interest in IF-set theory the old (new?) question has arisen: How to define and interpret inclusion between IF-sets? Although some measures for inclusion grade were suggested (see [7,13,28]), none of them had a natural and clear interpretation. Thus the motivation of the present paper is to propose a more natural tools for estimating the degree of inclusion between IF-sets and explore their properties.

The paper is organized as follows: In Section 2 we recall basic information on fuzzy sets and IF-sets. We also show there crisp definition of inclusion. In Section 3 we present some inclusion indicators for fuzzy sets and discuss briefly axioms for subsethood measures in standard fuzzy set theory. Then in Section 4 we suggest two inclusion indicators for IF-sets and examine their properties.

2. Fuzzy sets and IF-sets

A fuzzy set A in $\mathbb{X}$ is defined as a set of ordered pairs

$$A = \{\langle x, \mu_A(x)\rangle : x \in \mathbb{X}\}, \tag{3}$$

where $\mu_A : \mathbb{X} \to [0,1]$ is the membership function of A and $\mu_A(x)$ is the grade of belongingness of x into A. A family of all fuzzy sets in $\mathbb{X}$ will be denoted by $\mathbb{FS}(\mathbb{X})$.

According to Zadeh's seminal paper [25] introducing fuzzy sets we define inclusion for two fuzzy sets A and B in $\mathbb{X}$ as follows

$$A \subset_F B \iff \mu_A(x) \leqslant \mu_B(x) \quad \forall x \in \mathbb{X}, \tag{4}$$

where $\mu_A, \mu_B : \mathbb{X} \to [0,1]$ denote membership functions of the sets A and B, respectively. Thus, by Zadeh's definition, a fuzzy set A is a subset of a fuzzy set B if and only if the graph of $\mu_A(x)$ fits beneath the graph of $\mu_B(x)$ for all $x \in \mathbb{X}$.

Let us also recall that the kernel of $A \in \mathbb{FS}(\mathbb{X})$ is a (usual) set of all elements that surely belong to A, i.e.

$$ker(A) = \{x : x \in \mathbb{X}, \mu_A(x) = 1\}, \tag{5}$$

while the support of $A \in \mathbb{FS}(\mathbb{X})$ is the complement of the (usual) set of all elements that surely do not belong to A, i.e.

$$supp(A) = \{x : x \in \mathbb{X}, \mu_A(x) > 0\}, \tag{6}$$

or – in other words – the support is a set of all elements that possibly belong to A.

For each fuzzy set A the grade of nonbelongingness of x in A is automatically equal to $1 - \mu_A(x)$. However, in real life the linguistic negation does not always identify with logical negation. This situation is very common in natural language processing, computing with words and their applications in many area (see, e.g., preference selections [21], medical diagnosis [9], multicriteria and group decision making [17,18,22], pattern recognition [5,6,15] etc.). Thus although fuzzy set theory provides useful tools for dealing with uncertain information ([26,27]), Atanassov [1] suggested a generalization of classical fuzzy set, called an intuitionistic fuzzy set. To avoid possible misinterpretations with intuitionistic logic we call it further on an IF-set instead of intuitionistic fuzzy set (for a terminology discussion we refer the reader to [10,14]). Thus, an IF-set A in $\mathbb{X}$ is given by a set of ordered triples

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathbb{X}\}, \tag{7}$$

such that $\mu_A, \nu_A : \mathbb{X} \to [0,1]$ are functions satisfying a following condition

$$\mu_A(x) + \nu_A(x) \leqslant 1 \quad \forall x \in \mathbb{X}. \tag{8}$$

For each x the numbers $\mu_A(x)$ and $\nu_A(x)$ represent the degree of membership and degree of nonmembership of the element $x \in \mathbb{X}$ into A, respectively. A family of all IF-sets in $\mathbb{X}$ will be denoted by $\mathbb{IFS}(\mathbb{X})$.

The quantity

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \tag{9}$$

called the index of $A \in \mathbb{IFS}(\mathbb{X})$, quantifies the amount of indeterminacy associated with x in A. If for given $A \in \mathbb{IFS}(\mathbb{X})$ we have $\pi_A(x) = 0$ for every $x \in \mathbb{X}$ then, of course, $A \in \mathbb{FS}(\mathbb{X})$.

Since IF-set is a direct generalization of Zadeh's fuzzy set the definition of inclusion for IF-sets is strongly based on (4). Namely, if $A, B \in \mathbb{IFS}(\mathbb{X})$ then

$$A \subset_{IF} B \iff (\mu_A(x) \leqslant \mu_B(x) \quad \text{and } \nu_A(x) \geqslant \nu_B(x) \quad \forall x \in \mathbb{X}), \tag{10}$$

where $\mu_A, \mu_B : \mathbb{X} \to [0,1]$ denote membership functions while $\nu_A, \nu_B : \mathbb{X} \to [0,1]$ are nonmembership functions of the IF-sets A and B, respectively.

Many interesting operators have been defined in the family of all IF-sets (see Atanassov [2]). However, here we want to mention especially two operators called by Atanassov the necessity and possibility operators.

For any $A \in \mathbb{IFS}(\mathbb{X})$ the necessity operator $\Box : \mathbb{IFS}(\mathbb{X}) \to \mathbb{IFS}(\mathbb{X})$ is defined by

$$\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in \mathbb{X}\}, \tag{11}$$