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Quantum counting: Operator methods for discrete population dynamics with applications to cell division

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A B S T R A C T

The set of natural numbers may be identified with the spectrum of eigenvalues of an operator (quantum counting), and the dynamical equations of populations of discrete, countable items may be formulated using operator methods. These equations take the form of time dependent operator equations, involving Hamiltonian operators, from which the statistical time dependence of population numbers may be determined. The quantum operator method is illustrated by a novel approach to cell population dynamics. This involves Hamiltonians that mimic the process of stimulated cell division. We evaluate two different models, one in which the stimuli are expended in the division process and one in which the stimuli act as true catalysts. While the former model exhibits only bounded cell population variations, the latter exhibits two distinct regimes; one has bounded population fluctuations about a mean level and in the other, the population can undergo growth to levels that are orders of magnitude above threshold levels, through an instability that could be interpreted as a cancerous growth phase.

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1. Introduction

Mathematical models incorporating differential equations have been in common use to investigate the dynamical behaviour of populations of systems of living things ever since Lotka (1925) and Volterra (1926) introduced their model of predator-prey competition in the 1920s. The nature of these models is often heuristic and it is usually taken for granted that the number continuum on the real number line can be used to model systems of discrete, countable entities like people, animals, plants, bacteria, cells, etc. Ecological systems (DeAngelis, 2003), the spread of epidemics (Isea and Lonngren, 2016), and cancer cell population growth (Daukste et al., 2012; Isea and Lonngren, 2014) are just a few examples of what has been modelled in this way.

The continuum approach does lead to simplifications, since we can use continuous, scalar-valued functions and ordinary differential calculus for rates of change of such populations. This approximation is often justified by arguing that if one is only interested in averages, as is usually the case in population models, then real numbers and not just counting numbers, are justifiable in most cases, especially when large populations are involved. Then, also, the minimum change in population number, being one, is a small fraction of the population as a whole, so any errors incurred should be small. However, it remains unclear whether modelling the average is the same thing as averaging a model, in the case of natural number valued populations, especially when population numbers are not large. Bagarello (2013) has recently shown how the number operator that is widely used in quantum theory can also be used to model discrete populations in social science and ecological contexts, and has pioneered a new approach to population dynamics based on this idea. The method is particularly relevant to closed ecosystems (Bagarello and Oliveri, 2014), where conservation rules play an important role in constraining the dynamics.

Such an approach might well be considered far fetched when endeavouring to persuade non-physicists that quantum tools are relevant to situations like predator-prey competition, that do not involve the often counterintuitive behaviour of quantum phenomena. Furthermore, number operators and associated creation and annihilation operators that are ubiquitous in quantum field theory, especially where this deals with the many-body problem in condensed matter physics, were developed from the quantum theory of mechanical oscillators, as part of a procedure called *second quantization*. *First quantization* refers to the replacement of the scalar dynamical variables of classical physics by operators that operate on scalar wavefunctions. Second quantization refers to the procedure whereby the wavefunctions of the first quantization are themselves replaced by operators that are the primitive fields of quantum field theory. This approach to physical theory was developed by Dirac (1964) and others in the 1920s and 30s. However, there is in fact a strong analogy between identical particles in many-body quantum field theory and macroscopic systems of many individuals, where a detailed description of the individuals is unimportant, but where the number of individuals within defined categories is all the information that is needed to define and model such systems. One reason for this strong connection is that the set of natural numbers that represent discrete populations corresponds to the spectrum of the eigenvalues of an operator. Then operator valued calculus becomes the appropriate way of dealing mathematically with population dynamics. We refer to this as *quantum counting* (Robinson and Haven, 2015), because of the connection between operator valued variables and quantum physics.[1]

[1] The term *quantum counting* is also used in the context of quantum search algorithms in quantum computing (Grover, 1997).

The paper is set out as follows. In section 2, we review both the basic quantum operator formalism needed to represent discrete populations and also the Heisenberg representation of time dependence. Here we largely adopt Bagarello's (Bagarello, 2013) approach of importing the relevant algebra from quantum many-body physics. In section 3, we illustrate how operator formalism may be used to model the dynamics of interacting populations, using some simple examples. A comparison with a classical representation of two-category interaction in the form of the Lotka-Volterra predator-prey system is also presented in section 3. In section 4 we introduce a novel application of the general method to cell division and cell population dynamics. The results are summarised in section 5.

2. Operator methods for discrete population dynamics

2.1. Operators and states of a system

The basic representation of a system using operators may be summarised by the equation

$$\widehat{\Gamma}|\gamma\rangle = \gamma|\gamma\rangle,$$

where $\widehat{\Gamma}$ is an operator, operating on a state, $|\gamma\rangle$[2] that has an eigenvalue γ. In any representation of operators, γ is invariably simply a number that constitutes some information about the state $|\gamma\rangle$ of a system. The system in question does not need to be anything physical, just something that can be represented mathematically. For example, the system could be an electronic bank account with γ the amount of money it contains. Typically, one specifies the operator first and then one solves the eigenvalue equation for both γ and $|\gamma\rangle$, simultaneously. There are usually several solutions, implying that the system can be in several states, each with its own value (i.e., eigenvalue). These correspond to some information about the system that we can, in principle at least, obtain from some measurements on the system.

There are two common representations of operators in use in quantum mechanics. One involves differential operators, in which case the eigenstates are represented by functions of the variable with respect to which they are differentiated. The other uses square matrices to represent the operator, then the eigenstates are represented by column vectors. However, these may well be of infinite dimensions and great care needs to be exercised in their use.

For the systems we will be dealing with in the rest of this paper, the only information we need to describe them is the number of items they contain, so the eigenvalues we need are simply the set of natural numbers.[3] The only kind of measurement we need to carry out on such as system, in order to obtain the required information, is counting the number of items it contains. In the next section we summarize the basic properties of creation, annihilation and number operators for a system comprising a single category of countable individual items. These operators are then generalized for more elaborate multi-category systems in the subsequent sections.

2.2. Creation and annihilation operators and the number operator

We begin by defining a non-commuting pair of operators, $\widehat{a}$ and $\widehat{a}^{\dagger}$, where $\widehat{a}^{\dagger}$ is the adjoint of $\widehat{a}$.[4] Their non-commuting properties

[2] Here we use Dirac notation for states (Auletta et al., 2009).

[3] In what follows, we always include zero in the set of natural numbers.

[4] See Weinberg (2013) and also Bagarello (2013) for mathematical details on adjoints.