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Voting games with abstention: Linking completeness and weightedness ☆

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ABSTRACT

Weighted games for several levels of approval in input and output were introduced in [9]. An extension of the desirability relation for simple games, called the influence relation, was introduced for games with several levels of approval in input in [24] (see also [18]). However, there are weighted games not being complete for the influence relation, something different to what occurs for simple games. In this paper we introduce several extensions of the desirability relation for simple games and from the completeness of them it follows the consistent link with weighted games, which solves the existing gap. Moreover, we prove that the influence relation is consistent with a known subclass of weighted games: strongly weighted games.

1. Introduction

Voting systems in democratic institutions, as those in international economic organizations or federal voting bodies, have in common that voters must make decisions involving a choice between multiple alternatives instead of the most usual assumption which assumes that voters are only allowed to vote for "yes" or "no". The specific or definable decision context consists of "either breaking the status quo or not". The target system describes each situation in which partitions of voters are able to pass a new law or change the status quo. Making decisions in democratic organizations is regarded as a DSS and an investigation of the DSS literature reveals that research has mainly focused on the effects of design, implementation and use on decision outcomes (see e.g., [3,11]).

The generalization of simple voting games to multiple levels arose out of the observation that, while many real voting systems allow voters to abstain (or be absent), simple games, by their nature, cannot take this possibility into account; those who do not vote "yes" are presumed to vote "no". Some works that took more than two input alternatives into consideration are: [5,19,1,12,14,16,17].

The voting structures, we primarily consider in this paper, are particular cases of (j, k) voting systems introduced in [9]. These structures assume that levels of approval in both, input and output, are ordered. The paper is confined to the case $k = 2$ and is focused for $j = 3$ ordered levels of input approval, although the results obtained in this paper extend for any arbitrary greater value of j. When absent voters are taken into account with a quorum (like in [4] or in [25]) the levels of input approval are not ordered and therefore, the results in this paper do not extend to that context. Some $(3, 2)$ voting systems are weighted $(3, 2)$ systems which admit a representation by means of vector weights and a threshold for the system, and therefore their representations as weighted systems are useful to separate the two possible collective outcomes. A purpose of this paper is to link the completeness of some desirability relations that determine the importance of voters in the system, with weighted systems with several ordered levels of approval for the input.

A necessary but not sufficient condition for a simple game to be representable as a weighted game is to be complete, i.e., all players are pairwise comparable by the desirability relation, which is a pre-ordering on the set of voters and therefore a reflexive and transitive relation. Consequently, an easy and practical way to identify some non-weighted simple games is to check that they are not complete.

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The notion of weighted game for (j,k) simple games is supported by a powerful combinatorial argument, grade–trade robustness for partitions. Even the issue of ascertaining whether an anonymous $(j,2)$ game is weighted is a difficult issue [10,26]. When we are restricted to simple games, i.e. $j = k = 2$, weighted (j,k) games are simply weighted simple games and grade–trade robustness for partitions (see [9]) is trade-robustness for coalitions (see [21] and [23]).

In this paper, we will look at the extension of the desirability relation for simple games [13] to the ternary voting game (or more generally for (3, 2) games) given in [24], wherein such an extension was denominated influence relation. In [24], it is proved that the influence relation fails to be transitive and cycles for players are possible. We observe that one may easily find weighted (3, 2) games which are not complete for the influence relation, so that the completeness of the game for the influence relation is not a necessary condition for a (3, 2) game to be weighted. To solve this gap we consider three separate new relations, each of them weaker than the influence relation. Then the desired notion of completeness is derived by demanding the completeness of each one of these three new relations. This notion is weaker than the completeness for the influence relation, but it is enough to become a necessary condition for the (3, 2) game to be weighted. Moreover, we will prove that the completeness derived by the influence relation becomes a necessary condition for a (3, 2) game to be strongly weighted, a subclass of weighted games already considered in [9].

An additional issue is also considered in this paper. It concerns the associated notions of swap-robustness for each of the three new relations introduced in the paper. These characterizations extend the known characterization of complete simple games by swap-robustness given in [22] and [20].

The paper is organized as follows. The technical background as well as an example is introduced in what remains of this section. In Section 2 we introduce several notions of desirability for (3, 2) games and consider their completeness, their restrictions and extensions to simple games and the hierarchies they induce. In Section 3 we consistently link weighted (3, 2) games with an appropriate class of complete (3, 2) games, and strongly weighted (3, 2) games with complete (3, 2) games with respect to the influence relation. In Section 4 different (3, 2) swap robustness properties, which restriction for the case of simple games constitutes a characterization of complete games, are established for the derived notions of completeness for (3,2) games. Conclusion ends the paper.

1.1. The class of (3, 2) simple games

The material on this section is essentially taken from Freixas and Zwicker [9], where (j,k) simple games are introduced, for the particular choices: $j = 3$ and $k = 2$. Before the main notions are introduced we need some preliminary definitions. An *ordered tripartition* of the finite set N is a sequence $S = (S_1,S_2,S_3)$ of mutually disjoint sets whose union is N. Any S_i is allowed to be empty, and we think of S_i as the set of those voters of N who vote approval level i for the issue at hand (where approval level 1 is the highest level of approval, 2 is the intermediate level and 3 the lowest level). The most relevant situation that happens in voting is when S_1 corresponds to the set of "yes" voters, S_2 to the set of abstainers and S_3 to the set of "no" voters. Thus, an ordered tripartition is the analog of a coalition for a standard simple game. Let 3^N denote the set of all ordered tripartitions of N. For $S, T \in 3^N$, we write $S \subseteq^3 T$ to mean that either $S = T$ or S may be transformed into T by shifting 1 or more voters to higher levels of approval. This is the same as saying $S_1 \subseteq T_1$ and $S_1 \cup S_2 \subseteq T_1 \cup T_2$; we write $S \subset^3 T$ if $S \subseteq^3 T$ and $S \neq T$. The $\subseteq^3$ order defined on 3^N has minimum: the tripartition $\mathcal{N}$ such that $\mathcal{N}_3 = N$, and maximum: the tripartition $\mathcal{M}$ such that $\mathcal{M}_1 = N$; i.e., for every tripartition S, $\mathcal{N} \subseteq^3 S \subseteq^3 \mathcal{M}$ holds.

Definition 1.1. A (3, 2) simple game $G = (N,V)$ (henceforth (3, 2) game) consists of a finite set N of voters together with a value function $V : 3^N \to \{0,1\}$, which satisfies $V(\mathcal{N}) = 0$, $V(\mathcal{M}) = 1$, and is monotonic: for all ordered tripartitions S and T, if $S \subset^3 T$ then $V(S) \leq V(T)$.

A (3, 2) game is also defined by the set of winning tripartitions $W = \{S \in 3^N : V(S) = 1\}$ that satisfies $\mathcal{N} \notin W$, $\mathcal{M} \in W$, and the monotonicity requirement: if $S \subset^3 T$ and $S \in W$ then $T \in W$.

Standard notions for coalitions in simple games naturally extend for tripartitions in (3,2) games: S is a *losing* tripartition whenever $V(S) = 0$, let L denote the set of losing tripartitions; S is a *minimal* winning tripartition provided that S is winning and for all $T \in 3^N$ such that $T \subset^3 S$, T is losing, let W^m denote the set of minimal winning tripartitions; S is a *maximal* losing tripartition provided that S is losing and for all $T \in 3^N$ such that $S \subset^3 T$, T is winning, let L^M denote the set of maximal losing tripartitions. It is clear that W and L form a bipartition of 3^N, and that each of the sets: W, L, W^m, and L^M uniquely determines the (3, 2) game.

Definition 1.2. Let G = (N, V) be a (3, 2) game. A representation of G as a weighted (3, 2) game consists of a vector $w = (w_1,w_2,w_3)$ where $w_i : N \to \mathbb{R}$ for each i together with a real number quota q such that for every tripartition S, V(S) = 1 if and only if $w(S) \geq q$, where w(S) denotes

$$\sum_{i=1}^{3}\sum_{p \in S_i} w_i(p) \text{ and } w_1(p) \geq w_2(p) \geq w_3(p) \text{ for each } p \in N.$$

We say that G = (N, V) is a weighted (3, 2) game if it has such a representation.

As was observed in [9], each "yes" voter contributes the weight $w_1(p)$ to the total weight H; each abstainer contributes $w_2(p)$ to H, and each "no" voter contributes $w_3(p)$ to H, with the issue passing exactly if H meets or exceeds some preset quota q. That is, before any voting takes place each voter is pre-assigned three weights with $w_1(p) \geq w_2(p) \geq w_3(p)$ for each voter p, but will make no assumptions about the signs of $w_1(p)$, $w_2(p)$ or $w_3(p)$. As occurs for simple games where two weights represent superfluous information, three weights represent superfluous information. If we renormalize by subtracting $w_2(p)$ from each of the weights $w_1(p)$, $w_2(p)$ and $w_3(p)$ then the new triple of weights $w^+(p) = w_1(p) - w_2(p)$, 0, and $w^-(p) = w_3(p) - w_2(p)$ describes the same voting system, and satisfies $w^+(p) \geq 0 \geq w^-(p)$.

A stronger condition of a weighted (3, 2) game introduced in [9] is the following.

Definition 1.3. A strongly weighted (3, 2) game is a weighted (3, 2) game that admits a representation such that for every pair of voters p and r, either

$$w^+(p) \geq w^+(r) \text{ and } -w^-(p) \geq -w^-(r)$$

or

$$w^+(p) \leq w^+(r) \text{ and } -w^-(p) \leq -w^-(r).$$

Example 1.4. Consider the (3, 2) game with a set of voters $N = \{a,b,c\}$:

$$W^m = \{(a,b,c), (b,c,a), (c,a,b)\}.$$

From the set of minimal winning tripartitions one may easily generate the set of winning tripartitions, the set of losing tripartitions and the set of maximal losing tripartitions, which is:

$$L^M = \{(a,c,b), (b,a,c), (c,b,a), (\emptyset, abc, \emptyset)\}.$$