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Combining partially independent belief functions

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ABSTRACT

The theory of belief functions manages uncertainty and also proposes a set of combination rules to aggregate opinions of several sources. Some combination rules mix evidential information where sources are independent; other rules are suited to combine evidential information held by dependent sources. In this paper we have two main contributions: First we suggest a method to quantify sources' degree of independence that may guide the choice of the more appropriate set of combination rules. Second, we propose a new combination rule that takes consideration of sources' degree of independence. The proposed method is illustrated on generated mass functions.

1. Introduction

Uncertainty theories like the *theory of probabilities*, the *theory of fuzzy sets*[1], the *theory of possibilities*[2] and the *theory of belief functions*[3,4] model and manage uncertain data. The theory of belief functions can deal with imprecise and/or uncertain data provided by several belief holders and also combine them.

Combining several evidential information held by distinct belief holders aggregates their points of view by stressing common points. In the theory of belief functions, many combination rules are proposed, some of them like [2,5–9] are fitted to the aggregation of evidential information provided by cognitively independent sources whereas the *cautious*, *bold*[10] and *mean* combination rules can be applied when sources are *cognitively dependent*. The choice of combination rules depends on sources' independence.

Some researches are focused on doxastic independence of variables such as [11,12]; others [4,13] tackled cognitive and evidential independence of variables. This paper is focused on measuring the independence of sources and not that of variables. We suggest a statistical approach to estimate the independence of sources on the bases of all evidential information that they provide. The aim of estimating the independence of sources is to guide the choice of the combination rule to be used when combining their evidential information.

We propose also a new combination rule to aggregate evidential information and take into account the independence degree of their sources. The proposed combination rule is weighted with that degree of independence leading to the conjunctive rule [14] when sources are fully independent and to the cautious rule [10] when they are fully dependent.

In the sequel, we introduce in Section 2 preliminaries of the theory of belief functions. In the Section 3, an evidential clustering algorithm is detailed. This clustering algorithm will be used in the first step of the independence measure process. Independence measure is then detailed in Section 4. It is estimated in four steps: In the first step the clustering algorithm is applied. Second a mapping between clusters is performed; then independence of clusters and sources is deduced in the last two steps. Independence is learned for only two sources and then generalized for a greater number of sources. A new combination rule is proposed in the Section 5 taking into account the independence degree of sources. The proposed method is tested on random mass functions in Section 6. Finally, conclusions are drawn.

2. Theory of belief functions

The theory of belief functions was introduced by Dempster [3] and formalized by Shafer [4] to model imperfect data. The *frame of discernment* also called *universe of discourse*, $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$, is an exhaustive set of N mutually exclusive hypotheses ω_i. The *power set* 2^Ω is a set of all subsets of Ω; it is made of hypotheses and unions of hypotheses from. The *basic belief assignment* (BBA) commonly called *mass*

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function is a function defined on the power set 2^{Ω} and spans the interval [0,1] such that:

$$\sum_{A \subseteq \Omega} m(A) = 1. \tag{1}$$

A *basic belief mass* (BBM) also called *mass*, $m(A)$, is a degree of faith on the truth of A. The bbm, $m(A)$, is a degree of belief on A which can be committed to its subsets if further information justifies it [7].

Subsets A having a strictly positive mass are called *focal elements*. Union of all focal elements is called *core*. Shafer [4] assumed a normality condition such that $m(\emptyset) = 0$, thereafter Smets [14] relaxed this condition in order to tolerate $m(\emptyset) > 0$.

The frame of discernment can also be a focal element; its bbm, $m(\Omega)$, is interpreted as a degree of *ignorance*. In the case of *total ignorance*, $m(\Omega) = 1$.

A *simple support function* is a mass function with two focal elements including the frame of discernment. A simple support function m is defined as follows:

$$m(A) = \begin{cases} 1-w & \text{if } A = B \text{ for some } B \subset \Omega \\ w & \text{if } A = \Omega \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where A is a *focus* of that simple support function and $w \in [0,1]$ is its *weight*. A simple support function is simply noted as A^w. A nondogmatic mass function can be obtained by the combination of several simple support functions. Therefore, any nondogmatic mass function can be decomposed into several support functions using the *canonical decomposition* proposed by Smets [15].

The *belief function* (*bel*) is computed from a BBA m. The amount $bel(A)$ is the minimal belief on A justified by available information on B ($B \subseteq A$):

$$bel(A) = \sum_{B \subseteq A, B \neq} m(B). \tag{3}$$

The *plausibility function* (*pl*) is also derived from a BBA m. The amount $pl(A)$ is the maximal belief on A justified by information on B which are not contradictory with A ($A \cap B \neq \emptyset$):

$$pl(A) = \sum_{A \cap B \neq \emptyset} m(B). \tag{4}$$

Pignistic transformation computes pignistic probabilities from mass functions in the purpose of making a decision. The pignistic probability of a single hypothesis A is given by:

$$BetP(A) = \sum_{B \subseteq \Omega, B \neq \emptyset} \frac{|B \cap A|}{|B|} \frac{m(B)}{1-m(\emptyset)}. \tag{5}$$

Decision is made according to the maximum pignistic probability. The single point having the greatest *BetP* is the most likely hypothesis.

2.1. Discounting

Sources of information are not always reliable, they can be unreliable or even a little bit reliable. Taking into account reliability of sources, we adjust their beliefs proportionally to degrees of reliability. Discounting mass functions is a way of taking consideration of sources' reliabilities into their mass functions. If reliability rate α of a source is known or can be quantified; discounting its mass function m is defined as follows:

$$\begin{cases} m^{\alpha}(A) & = \alpha \times m(A) \\ m^{\alpha}(\Omega) & = 1\text{-}\alpha \times (1\text{-}m(\Omega)) \end{cases}, \forall A \subset \Omega \tag{6}$$

This discounting operator can be used not only to take consideration of source's reliability, but also to consider any information which can be integrated into the mass function, $(1 - \alpha)$ is called *discounting rate*.

2.2. Combination rules

In the theory of belief functions, a great number of combination rules are used to summarize a set of mass functions into only one. Let s_1 and s_2 be two distinct and cognitively independent sources providing two different mass functions m_1 and m_2 defined on the same frame of discernment Ω. Combining these mass functions induces a third one m_{12} defined on the same frame of discernment Ω.

There is a great number of combination rules [2,5–9], but we enumerate in this section only Dempster, conjunctive, disjunctive, Yager, Dubois and Prade, mean, cautious and bold combination rules. The first combination rule was proposed by Dempster in [3] to combine two distinct mass functions m_1 and m_2 as follows:

$$m_{1\oplus 2}(A) = (m_1 \oplus m_2)(A) = \begin{cases} \dfrac{\sum_{B \cap C = A} m_1(B) \times m_2(C)}{1 - \sum_{B \cap C = \emptyset} m_1(B) \times m_2(C)} & \forall A \subseteq \Omega, A \neq \emptyset \\ 0 & \text{if } A = \emptyset \end{cases}. \tag{7}$$

The BBM of the empty set is null ($m(\emptyset) = 0$). This rule verifies the normality condition and works under a *closed world* where Ω is exhaustive.

In order to solve the problem highlighted by Zadeh's counter example [16] where Dempster's rule of combination produced unsatisfactory results, many combination rules appeared. Smets [14] proposed an *open world* where a positive mass can be allocated to the empty set. Hence the conjunctive rule of combination for two mass functions m_1 and m_2 is defined as follows:

$$m_{1 \textcircled{\cap} 2}(A) = (m_1 \textcircled{\cap} m_2)(A) = \sum_{B \cap C = A} m_1(B) \times m_2(C) \tag{8}$$

Even if Smets [17] interpreted the bbm, $m_1 m_{1 \textcircled{\cap} 2}(\emptyset)_2(\emptyset)$, as an amount of conflict between evidences that induced m_1 and m_2; that amount is not really a conflict because it includes a certain degree of auto-conflict due to the non-idempotence of the conjunctive combination [18].

The conjunctive rule is used only when both sources are reliable. Smets [14] proposed also to use a disjunctive combination when an unknown source is unreliable. The disjunctive rule of combination is defined for two bbas m_1 and m_2 as follows:

$$m_{1 \textcircled{\cup} 2}(A) = (m_1 \textcircled{\cup} m_2)(A) = \sum_{B \cup C = A} m_1(B) \times m_2(C) \tag{9}$$

Yager in [8] interpreted $m(\emptyset)$ as an amount of ignorance; consequently it is allocated to Ω. Yager's rule of combination is also defined to combine two mass functions m_1 and m_2 as follows:

$$\begin{cases} m_Y(X) = m_{1 \textcircled{\cap} 2}(X) & \forall X \subset \Omega, X \neq \emptyset \\ m_Y(\Omega) = m_{1 \textcircled{\cap} 2}(\Omega) + m_{1 \textcircled{\cap} 2}(\emptyset) \\ m_Y(\emptyset) = 0 \end{cases} \tag{10}$$