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Convex cone-based partial order for multiple criteria alternatives☆,☆☆

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ABSTRACT

In this paper, we consider the problem of finding a preference-based strict partial order for a finite set of multiple criteria alternatives. We develop an approach based on information provided by the decision maker in the form of pairwise comparisons. We assume that the decision maker's value function is not explicitly known, but it has a quasi-concave form. Based on this assumption, we construct convex cones providing additional preference information to partially order the set of alternatives. We also extend the information obtained from the quasi-concavity of the value function to derive heuristic information that enriches the strict partial order. This approach can as such be used to partially rank multiple criteria alternatives and as a supplementary method to incorporate preference information in, e.g. Data Envelopment Analysis and Evolutionary Multi-Objective Optimization.

1. Introduction

The purpose in Multiple Criteria Decision Making (MCDM) is to find the most preferred solution among a set of implicitly or explicitly defined alternatives characterized by several criteria, or to rank such alternatives. The problems where alternatives are implicitly defined using constraints are called multiple criteria design problems and the problems where alternatives are explicitly given are called multiple criteria evaluation problems. In this paper, we consider multiple criteria evaluation problems where a Decision Maker (DM) evaluates the explicitly given alternatives.

Which kind of approach is most suitable to solving evaluation problems is heavily dependent on the characteristics of the problem. The outranking approach [21], the multi-attribute value function approach [6], the analytic hierarchy process [22], the regime method [4], the hierarchical interactive approach [7], the visual reference direction approach [8], the aspiration-level interactive method (AIM) [18,19], and the hybrid method [17] are typical examples of approaches developed to solve evaluation problems.

A class of methods is based on implicitly known value functions. No attempt is made to construct the value function, but assumptions of its functional form are used to structure the search process. Typical assumptions are linearity, Chebyshev-type min–max function, quasi-concavity, pseudo-concavity, etc. of the value function. Examples of such methods are presented in Refs. [10–13,15,26]. There are also approaches to find which form of value function the DM's preferences are consistent with [14,24].

Various interaction styles have been proposed for interactive approaches in general. Examples include requiring pairwise comparison of alternatives [27], local tradeoff ratios [2], interval local tradeoff ratios [23], comparative tradeoff ratios [5], reference points [25], and reference directions [9]. A good interactive approach does not waste the DM's time, and its communication language is easy. Furthermore, it is a good idea to increase the intelligence of the system, but it is important to remember that the DM wants to keep the control of the system in his/her own hands. There are several ways to implement a dialogue between an interactive approach and the DM. In this paper we require pairwise comparison information as we think it is easy and relevant for a DM to compare pairs of alternatives.

Our aim in this paper is to produce a preference-based strict partial order for a finite set of multiple criteria alternatives. We try to create the strict partial order by making maximum use of the available preference information in the form of pairwise comparisons. We assume that the DM's value function is unknown to us, but it has a quasi-concave form. Based on the available preference information and exploiting the implications of a quasi-concave value function, we construct convex cones [10] and polyhedrons to provide additional preference relations that enrich the strict partial order of alternatives. We also introduce heuristics to extract further approximate preference relations that can be used in the partial order.

This paper unfolds as follows. Section 2 provides preliminary considerations. Section 3 develops the main idea and formulations.

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Section 4 includes an illustrative example and finally Section 5 concludes the paper.

2. Preliminary considerations

Consider a discrete, finite, deterministic multiple criteria evaluation problem where a single DM compares a set of n alternatives with respect to p criteria. The set S of alternatives includes vectors $X_i \in \Re^p$, $i \in N = \{1, \ldots, n\}$, with elements $x_{ij} > 0$ for all j. Without loss of generality, assume that for each criterion more is better. We define the dominance in $\Re^p$ in the usual way.

Definition 1. A vector $X^* \in \Re^p$ is non-dominated iff (if and only if) there does not exist another $X \in \Re^p$ such that $X \geq X^*$ and $X \neq X^*$.

Definition 2. The function f: $\Omega \rightarrow \Re$, $\Omega \subseteq \Re^p$, is called a value function if it has the following properties:

1. $f(X^*) > f(X)$, if X^* dominates X.
2. $f(X^*) > f(X)$, iff X^* is preferred to X.
3. $f(X^*) \geq f(X)$, if X^* is at least as preferred as X.

Property 1 implies that function f is also strictly increasing in set Ω, where Ω consists of all points at which the value function evaluation is needed. Hence forward, we assume that the DM's value function is quasi-concave and that we only know of its form.

In the following we use the symbol "$\succ$" to indicate the relationship "is preferred to." When needed, we also use the symbol "$\succeq$" to indicate "is at least as good as." It can be seen that both relations are transitive. The DM's preferences are expressed by the set $P = \{(X_r, X_s) | X_r \succ X_s, r, s \in N\}$. Thus P defines a strict partial order in S (an asymmetric transitive binary relation over S).

The efficiency of interactive procedures is heavily dependent on what kind of and how much information the DM is required to provide. The assumptions that are made about the value function facilitate the convergence of an interactive procedure. However, such assumptions should be as realistic as possible. For example, a linear value function assumption produces a fast convergence, but none of the convex dominated alternatives can be most preferred [28]. The quasi-concavity assumption of the value function is quite general, yet powerful in constructing a strict partial order for alternatives. Moreover, convex dominated alternatives can be most preferred. When assuming that the value function is quasi-concave, based on pairwise preference information, we may construct so called convex cones, characterizing the vertex of the cones by inferior alternatives.

To be more precise, assume that we have m (distinct) points $X_1, \ldots, X_{k-1}, X_k, \ldots, X_m$ such that $X_i \succ X_k$ for $i = 1, \ldots, m$ and $i \neq k$. Then by Definition 2, $f(X_k) < f(X_i)$, $i = 1, \ldots, m$, and $i \neq k$. The subset of S including m different points $X_1, \ldots, X_{k-1}, X_k, \ldots, X_m$ such that $X_i \succ X_k$ for $i = 1, \ldots, m$ and $i \neq k$ is called a *preference subset* and is denoted by $\{X_1, \ldots, X_m; X_k\}$.

Based on this preference subset, we may construct a cone where alternative X_k is the vertex of the cone. We define the cone $C(X_1, \ldots, X_m; X_k)$ with vertex X_k as follows:

$$C(X_1, \ldots, X_m; X_k) = \{X | X = X_k + \sum_{i \neq k} \mu_i (X_k - X_i), \mu_i \geq 0, i = 1, \ldots, m, i \neq k\}.$$

Based on the quasi-concavity assumption of our value function f, for any $Z \in C(X_1, \ldots, X_m; X_k)$, we have shown ([10]) that $f(X_i) > f(X_k) \geq f(Z)$ for $i = 1, \ldots, m$ and $i \neq k$ which means that $X_k \succeq Z$. Each point $Z \in C(X_1, \ldots, X_m; X_k)$, $Z \neq X_k$, or any point V dominated by Z, is called *cone dominated*.

Moreover, we may define a polyhedron spanned by the points $X_1, \ldots, X_m$ as follows:

$$H(X_1, \ldots, X_m) = \{X | X = \sum_{i=1}^{m} \mu_i X_i, \sum_{i=1}^{m} \mu_i = 1, \mu_i \geq 0, i = 1, \ldots, m\}.$$

If $Y \in H(X_1, \ldots, X_m)$, then from the definition of quasi-concavity it follows that $f(Y) = f\left(\sum_{i=1}^{m} \mu_i X_i\right) \geq \min_i f(X_i) = f(X_k)$ which means that $Y \succeq X_k$.

In fact for each preference subset $\{X_1, \ldots, X_m; X_k\}$ we may define a convex cone and a polyhedron. Based on the structure of these two sets and the assumptions made about the value function, it is possible to extract more preference information about other points not belonging to the corresponding preference set with respect to X_k by checking whether or not the point belongs to the convex cone $C(X_1, \ldots, X_m; X_k)$ or falls under it, or belongs to the polyhedron $H(X_1, \ldots, X_m)$ or lies above it.

Without loss of generality, let the DM's preference set P be expressed as the union of several preference subsets—each one characterized by its worst point as the vertex. Different preference subsets yield different convex cones and polyhedrons which enable us to extract more information to update the preference set P by adding new preference comparisons.

In Fig. 1, we illustrate how the quasi-concavity assumption brings additional information to the preference-based ranking of alternatives.

Consider alternative X_2. In case we have no preference information, we can use only dominance. For any Y_1 in the region dominating X_2(region E_3 in Fig. 1) we have $Y_1 \geq X_2$, $Y_1 \neq X_2$ which implies that $Y_1 \succ X_2$. Similarly, for any Y_2 in the region dominated by X_2(region A_1 in Fig. 1) we have $X_2 \geq Y_2$, $X_2 \neq Y_2$ which implies that $X_2 \succ Y_2$. Finally, if $(X_1, X_2) \in P$ then for any Y_3 in the region dominating X_1(region E_2 in Fig. 1), we have $Y_3 \succ X_1 \succ X_2$.

When we use the quasi-concavity assumption, we can exploit the available preference information further, as we will show in detail in the next section. Specifically, we may use a convex cone to conclude that for any Y_4 in region A_2 in Fig. 1 we have $X_2 \succeq Y_4$ and we may use a polyhedron to conclude that for any Y_5 in region E_1 in Fig. 1 we have $Y_5 \succeq X_2$.

3. The method

The goal is to implement the above concepts from [12] to define a strict partial order for set S. Moreover, Prasad et al. [20] proposed an approach extending the idea of convex cones heuristically. They developed the concept of p-cone efficiency, providing a measure to find out how close an alternative is from being dominated by the cone under consideration. The smaller the measure, the closer the alternative is to being dominated.

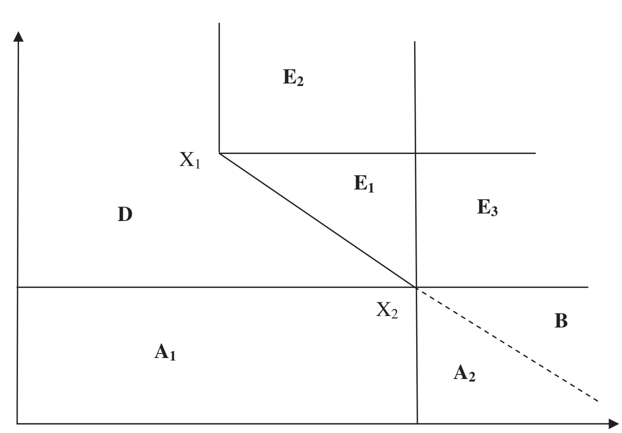

Fig. 1. Illustration of additional information provided by quasi-concavity.