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The significance threshold for coherence when using the Welch's periodogram method: Effect of overlapping segments

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ABSTRACT

A classical tool to identify coupling mechanisms between two biological signals is transfer function (TF) analysis, commonly estimated by the Welch's periodogram method. However, the reliability of TF estimation depends on the coherence function, an index of linear coupling in the frequency domain. Thus, a threshold value has to be determined before assuming a coupling between two signals at a given frequency.

Koopmans (*The spectral analysis of time series*) proposed a theoretical expression in the case of non-overlapping segments. Overlapping segments, however, improves the estimation of TF.

In the present study, a modification of the expression by Koopmans is proposed for taking into account the effect of the overlapping procedure, including the most common case of a 50% overlapping ratio. Simulation tests were done on non-coherent signals (using the Hanning window), and a Kolmogorov–Smirnov test was used to check for the adequacy with the proposed threshold formula. The results showed that for an overlapping ratio of 50%, no statistical difference could be found ($P > 0.05$), except when using less than seven segments.

The method has been applied to the computation of the transfer function between systolic blood pressure and heart rate.

1. Introduction

The coherence function $\gamma^2(f)$ is used as an index of correlation in the frequency domain between two signals $x(t)$ and $y(t)$. It is usually estimated by using the Welch's periodogram method [1]. With this method, signals are split into overlapping windowed segments, spectra are computed for each segment with a Fourier transform, and averaged.

If $\gamma^2(f)$ is different from zero at a certain frequency, then one signal can be represented to a certain degree as a linear function of the other. However, as the method provides an estimate $\hat{\gamma}^2(f)$ of the true value $\gamma^2(f)$, the assumption that $\gamma^2(f)$ is strictly positive carries a certain risk α of being wrong. This leads to the definition of the threshold value $\gamma^2_{th}(\alpha)$ as

$$P[\hat{\gamma}^2(f) > \gamma^2_{th}(\alpha) | \gamma^2(f) = 0] = \alpha. \tag{1}$$

Namely, if $\hat{\gamma}^2(f)$ is above the threshold, then $\gamma^2(f)$ exceeds zero at the significance level α.

A threshold value of 0.5 proposed by de Boer et al. [2] has been widely used [3–5]. The justification for using a value of 0.5 is that it points to 50% of shared variance between the signals.

Koopmans [6] derived an approximation of the distribution of $\hat{\gamma}^2(f)$ when $\gamma^2(f) = 0$ for Gaussian signals in the case of non-overlapping segments, which has been used in some studies [7–9]. However, as initially proposed by Welch [1], overlapping segments efficiently reduces the variance of the spectral estimates by using a larger part of the signal. But, because the independence between segments is lost, the distribution of $\hat{\gamma}^2(f)$ is no longer the same. Different approaches have been followed to deal with this problem.

Some authors [10] chose to consider only independent segments, which provides a significance threshold for coherence always higher than the true one, so that non-coherent values are not falsely detected. Taylor et al. [11] used the same approach but incorporated a correction factor to take into account the overlapping effect. However, these authors did not give any details about the computation of this factor.

Gross et al. [12] chose to compute the coherence function between two signals that were shifted in time to nullify the true coherence, and they took the threshold at level α as the $1-\alpha$ quantile of all coherence values. However, inaccurate results were reported with this method [13]. A similar approach has been presented using a surrogate data analysis, where the two signals

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are randomly shuffled to eliminate any correlation between them [14].

Finally, Bortel and Sovka [13] proposed a formula in the case of a maximum overlapping, i.e. when the segments are shifted by one value only. Because of the decimation-in-time property of the Fourier transform, the spectra can be estimated using a lower overlapping ratio, down to a certain value which depends on the window [15]. In the case of the Hanning window, the overlapping ratio can be reduced down to 75% without altering the coherence distribution [13]. It is not known whether the method can be used for lower values of the overlapping ratio, including the 50% ratio, which is by far the most commonly used ratio in biomedical research [10,16,17].

The aim of the present study was, therefore, to develop an extension of Koopmans' method [6] allowing the determination of the significance threshold for coherence whatever the overlapping ratio.

2. Theory

2.1. Introduction

Let $x(t)$, $y(t)$ be two stationary signals of length N. We define S_{xx} and S_{yy} as the spectral density functions of the two signals and S_{xy} as the cross-spectral density function.

We hypothesize that $y(t)$ is the output of a linear time-invariant filter L with input $x(t)$, contaminated additively by uncorrelated noise $n(t)$. As L is solely determined by its transfer function, we can define the filter $\tilde{L}$ whose transfer function is:

$$H(f) = \frac{S_{xy}(f)}{S_{xx}(f)}. \tag{2}$$

$\tilde{L}$ is the filter that minimizes the mean square error $E[(n(t))^2]$ [6, p. 140].

The coherence function is defined as:

$$\gamma_{xy}^2(f) = \frac{|S_{xy}(f)|^2}{S_{xx}(f)S_{yy}(f)}. \tag{3}$$

The spectral density function of $\tilde{L}(x)(t)$ can be expressed as $S_{\tilde{L}(x)\tilde{L}(x)} = |H(f)|^2 S_{xx}$. Using (2) and (3) gives $\gamma_{xy}^2(f) = S_{\tilde{L}(x)\tilde{L}(x)}(f)/S_{yy}(f)$, which can be described as the proportion of the power in $y(t)$ at frequency f which can be explained by the output of the filter $\tilde{L}$.

2.2. Welch's periodogram

The estimates $\hat{S}_{xy}(f)$, $\hat{S}_{xx}(f)$ and $\hat{S}_{yy}(f)$ of the spectra are computed using the Welch's periodogram method. Each signal is decomposed into K segments of length L, shifted by a fixed delay D:

$$\begin{aligned} x_k(t) &= x(kD + t) \\ y_k(t) &= y(kD + t) \\ t = 0, \ldots, L-1 &\quad k = 0, \ldots, K-1 \end{aligned} \tag{4}$$

so that $\delta = 100(1 - D/L)\,\%$ corresponds to the overlapping ratio.

Each segment is multiplied by a window function $w_L(t)$, $t = 0, \ldots, L-1$ and the spectra are computed using a Fourier transform

$$\begin{aligned} \hat{S}_{xx}(f) &= \frac{1}{U_{w_L}K}\sum_{k=0}^{K-1}|\mathcal{F}(x_k w_L)(f)|^2 \\ \hat{S}_{yy}(f) &= \frac{1}{U_{w_L}K}\sum_{k=0}^{K-1}|\mathcal{F}(y_k w_L)(f)|^2 \\ \hat{S}_{xy}(f) &= \frac{1}{U_{w_L}K}\sum_{k=0}^{K-1}\mathcal{F}(x_k w_L)(f)\overline{\mathcal{F}(y_k w_L)(f)}. \end{aligned} \tag{5}$$

with

$$U_{w_L} = \sum_{t=0}^{L-1} w_L^2(t) \tag{6}$$

The transfer function $\hat{H}(f)$ and the coherence function $\hat{\gamma}^2(f)$ can be estimated by substituting the spectra with their estimates in (2) and (3).

2.3. Distribution of the coherence

All the following distribution functions are approximations of the true ones. Also, the signals are considered as coming from stationary Gaussian processes.

Koopmans demonstrated that the auto-spectra follow a chi-square distribution with $2K$ degrees of freedom in the case of independent segments. Then, using the Wishart approximation for the distribution of the cross-spectrum, he expressed the value of the coherence threshold $\gamma_{th}^2(\alpha)$ as

$$\frac{(K-1)\gamma_{th}^2(\alpha)}{1-\gamma_{th}^2(\alpha)} = F_{2,2K-2}(\alpha), \tag{7}$$

where F is the Fisher distribution with 2 and $2K - 2$ degrees of freedom [6, p. 284]. Thus, the threshold value decreases as the number of segments increases. Eq. (7) can be simplified as

$$\gamma_{th}^2(\alpha) = 1 - \alpha^{1/(K-1)}, \tag{8}$$

The corresponding cumulative distribution function (CDF) F_K for $\hat{\gamma}^2(f)$ when $\gamma^2(f) = 0$ is

$$F_K(\hat{\gamma}^2(f)) = 1 - (1 - \hat{\gamma}^2(f))^{K-1}. \tag{9}$$

In the case of overlapping segments, Welch [1] approximates the relative variance of the auto-spectrum $\hat{S}(f)$ as

$$\frac{\mathrm{Var}[\hat{S}(f)]}{E[\hat{S}(f)]^2} = \frac{c_w(D)}{K} \tag{10}$$

$$c_w(D) = 1 + 2\sum_{j=1}^{K}\frac{K-j}{K}\rho_w^2(jD) \tag{11}$$

$$\rho_w(M) = \frac{\sum_{t=0}^{L-M-1} w_L(t)w_L(t+M)}{\sum_{t=0}^{L-1} w_L^2(t)} \tag{12}$$

However, for a chi-square random variable χ_r^2 with r degrees of freedom, we have $E[\chi_r^2] = r$ and $\mathrm{Var}[\chi_r^2] = 2r$, then the relative variance is

$$\frac{\mathrm{Var}[\chi_r^2]}{E[\chi_r^2]^2} = \frac{2}{r} \tag{13}$$

So, $\hat{S}(f)$ follows a chi-square distribution with $2K/c_w(D) = 2\tilde{K}$ equivalent degrees of freedom.

Our hypothesis is that in the case of overlapping segments, the threshold level and the CDF are obtained by replacing K with $\tilde{K}$ in (8) and (9). The proposed threshold level is then:

$$\gamma_{th}^2(\alpha) = 1 - \alpha^{1/(\tilde{K}-1)} \tag{14}$$

3. Testing

3.1. Kolmogorov–Smirnov test

Simulation tests were performed to check for the accuracy of the CDF $F_{\tilde{K}}$. Two gaussian white noise signals were generated and split into K segments of 128 points, overlapping by $\delta = 0\%$, 50% and