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Noise enhanced binary hypothesis-testing in a new framework

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ABSTRACT

In this paper, the noise enhanced system performance in a binary hypothesis testing problem is investigated when the additive noise is a convex combination of the optimal noise probability density functions (PDFs) obtained in two limit cases, which are the minimization of false-alarm probability (P_{FA}) without decreasing detection probability (P_D) and the maximization of P_D without increasing P_{FA}, respectively. Existing algorithms do not fully consider the relationship between the two limit cases and the optimal noise is often deduced according to only one limit case or Bayes criterion. We propose a new optimal noise framework which utilizes the two limit cases and deduce the PDFs of the new optimal noise. Furthermore, the sufficient conditions are derived to determine whether the performance of the detector can be improved or not via the new noise. In addition, the effects of the new noise are analyzed according to Bayes criterion. Rather than adjusting the additive noise again as shown in other algorithms, we just tune one parameter of the new optimal noise PDF to meet the different requirements under the Bayes criterion. Finally, an illustrative example is presented to study the theoretical results.

1. Introduction

From the perspective of the classical signal processing theory, noise is generally regarded as disturbances to the system [1]. In addition, the weak signal is often submerged in unwanted noises. Therefore, making use of various methods to filter out the noise is an efficient measure to enhance information transmission and improve system detection performance.

Nevertheless, noise does not always play a negative role for all signals and systems. On the contrary, the positive effect of noise has been paid widespread concern by researchers since Benzi first proposed the concept of stochastic resonance (SR) [2–9]. Some of the more representative studies of SR in signal and information processing are shown as below: a nonlinear bistable autoregressive model applied in signal detection by S. Zozor [4,5], the noise enhanced detection performance under the Neyman–Pearson criterion analyzed by H. Chen [7,8], etc.

Naturally, the phenomenon of noise enhanced system performance, i.e. SR, can also be found in signal detection. Researches in recent years indicate that the improvement of some nonlinear systems can be obtained by increasing the background noise level or injecting additive noise to their input [9–18]. The metric of improvements can be measured by the increase of output signal-to-noise (SNR) [9] or mutual information [10], or the decrease in the average error rate [11], etc.

In the hypothesis testing problem, the research on how to improve the performance of a suboptimal detector via adding an independent additive noise to their observations is usually based on Neyman–Pearson [9,12], Bayes [13,14] and Minimax criteria [15]. In fact, the noise enhancements can also be obtained for an optimal detector in some cases, as investigated in [5,12,16]. Such as, the noise enhanced performance of optimal Neyman–Pearson, Bayes and Minimax detectors are studied in [16], which proves that the performance of optimal detectors can be improved by increasing noise level of system under certain conditions.

According to Neyman–Pearson criterion, the purpose is to increase the detection probability under the constraint on false-alarm probability [7,8,12]. In [7], a mathematical framework to analyze the noise enhanced effect in binary hypothesis testing problems is developed based on Neyman–Pearson criterion. The corresponding mechanism for noise enhanced signal detection is explored. Sufficient conditions for improvability or non-improvability and the form of the optimal noise PDF as a randomization of at most two discrete vector are determined at the same time. At the end of that paper, an illustrative example is presented to compare the noise enhanced performance between different detectors where the optimal stochastic resonance noise, as well as Gaussian, uniform, and optimal symmetric noises are applied. The study in [7] is extended to restricted Neyman–Pearson framework in [17] and [18], where the constraints become more strictly. The aim in [18] is to maximize the average detection probability under

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the constraint on false-alarm and worst-case detection probabilities.

In the Bayes framework, the prior information is known, while it is unknown in the Minimax framework, the purpose of them is to minimize the Bayes risk. Besides, the restricted Bayes criterion is well suited in the case of prior information with some uncertainty which is most common in practical life. In [13], the effects of additive independent noise on the performance of some suboptimal detectors are investigated according to restricted Bayes criterion. In other words, the aim is to find the optimal noise which can minimize the Bayes risk under certain constraints on the conditional risk. Based on a generic M-ary composite hypothesis-testing formulation, the optimal probability distribution of additive noise and the sufficient conditions which the performance of a detector can or cannot be improved via additive noise are derived. Besides, it studies the simple hypothesis-testing problems in more detail, such as the solution of the optimization problem of additive noise for binary case. It is obvious that the noise enhanced problem in Bayes framework and Minimax framework [15] can be viewed as special cases in restricted Bayes framework.

Based on the analysis of articles related to Bayes, we have noticed an interesting phenomenon that people always regard the Bayes risk as a whole in binary hypothesis testing problems and ignore it can be expanded as a sum of a constant and a linear combination of detection and false-alarm probabilities. These methods can reduce the Bayes risk, but may not guarantee false alarm and detection performance of detector improved at the same time. If the optimal solution of Bayes risk is deduced by the optimization of detection and false-alarm probabilities directly, the problem can be solved more easily in some cases. Based on this conception, the minimization of Bayes risk is equivalent to the decrease of false alarm probability and the increase of detection probability simultaneously.

However, it is hard to find the optimal additive noise directly to satisfy the conditions that false alarm probability reduced and detection probability increased after adding the additive noise. Hence, we first find the optimal noises minimize false alarm probability without decreasing detection probability and maximize detection probability without increasing false alarm probability, respectively, then deduce the optimal noise for the minimum Bayes risk through the linear weighted combination of the two optimal noises.

Most researchers have focused mainly on improving the detection probability. For example, the case of maximizing detection probability without increasing false alarm probability is researched in [7,8]. But so far, no study has involved in minimizing the false alarm probability without decreasing the detection probability, which is also very important in binary hypothesis testing problems. Therefore, the aim of this paper includes how to minimize the false alarm probability without decreasing the detection probability via independent additive noise, obtain the corresponding optimal noise and derive the sufficient conditions. Furthermore, we find the optimal noise which can decrease false alarm probability and increase detection probability simultaneously, derive the corresponding sufficient conditions. More importantly, the minimization of Bayes risk can be achieved according to the analysis of false-alarm and detection probabilities above. The main contributions of this paper can be summarized as follows:

- Formulation of the noise enhanced model, which can increase the detection probability and decrease the probability of false-alarm at the same time.
- Determination of the optimal additive noise to minimize the false-alarm without decreasing detection probability, which is a randomization of two discrete vectors.
- Derivation of the suitable additive noise corresponding to our noise enhanced model, which is a suitable randomization of no more than four constant vectors obtained through the convex combination between the optimal noise PDFs of two limit cases.
- Derivation of the sufficient conditions for two limit cases and our noise enhanced model.
- Determination of the additive noise and the corresponding PDF which can minimize the Bayes risk according to the Bayes and Minimax criteria in certain conditions based on the analysis of the noise enhanced model.

The remainder of this paper is organized as follows. In Section 2, the binary hypothesis test problem formulation is given and a noise enhanced model based on the improvements of both the false-alarm and detection performance is proposed. The corresponding optimal additive noise and sufficient conditions for different cases are derived in Section 3. Then, the minimization of Bayes risk according to the Bayes and Minimax criteria under certain conditions is researched in Section 4. Finally, a detection example and the experiment results are presented in Section 5 and conclusions are made in Section 6.

2. Noise enhanced model in binary hypothesis testing

2.1. Problem formulation

Consider the binary hypothesis-testing problem as below:

$$H_i : p_i(\mathbf{x}), i = 0, 1, \tag{1}$$

where $p_i(\mathbf{x})$ represents the PDF of $\mathbf{x}$ under H_i, and $\mathbf{x}$ is a K-dimensional observation vector, i.e., $\mathbf{x} \in \mathbb{R}^K$. The decision rule is usually expressed as $\phi(\mathbf{x})$, which represents the probability of selecting H_1, $0 \le \phi(\mathbf{x}) \le 1$. As a result, the probabilities of false alarm and detection are provided by

$$P_{FA}^{x} = \int_{\mathbb{R}^K} \phi(\mathbf{x}) p_0(\mathbf{x}) d\mathbf{x}, \tag{2}$$

$$P_{D}^{x} = \int_{\mathbb{R}^K} \phi(\mathbf{x}) p_1(\mathbf{x}) d\mathbf{x}. \tag{3}$$

The noise modified observation $\mathbf{y}$ is given by adding an independent noise $\mathbf{n}$ to $\mathbf{x}$, i.e., $\mathbf{y} = \mathbf{x} + \mathbf{n}$. Then the PDF of $\mathbf{y}$ under H_i can be expressed by the convolutions of $p_i(\mathbf{x})$ and $\phi(\mathbf{x})$ as follows,

$$p_i(\mathbf{y}) = p_{\mathbf{n}}(\cdot) * p_i(\mathbf{x}) = \int_{\mathbb{R}^K} p_{\mathbf{n}}(\mathbf{n}) p_i(\mathbf{y} - \mathbf{n}) d\mathbf{n}. \tag{4}$$

When the detector is fixed, the decision function of $\mathbf{y}$ is the same as that of $\mathbf{x}$, namely $\phi(\mathbf{y}) = \phi(\mathbf{x})$. Therefore, the new probabilities of false alarm and detection for $\mathbf{y}$ can be shown as below,

$$\begin{aligned} P_{FA}^{y} &= \int_{\mathbb{R}^K} \phi(\mathbf{y}) p_0(\mathbf{y}) d\mathbf{y} = \int_{\mathbb{R}^K} p_{\mathbf{n}}(\mathbf{n}) \Big(\int_{\mathbb{R}^K} \phi(\mathbf{y}) p_0(\mathbf{y} - \mathbf{n}) d\mathbf{y} \Big) d\mathbf{n} \\ &= \int_{\mathbb{R}^K} p_{\mathbf{n}}(\mathbf{n}) F_0(\mathbf{n}) d\mathbf{n} = E_{\mathbf{n}}(F_0(\mathbf{n})), \end{aligned} \tag{5}$$

$$P_{D}^{y} = \int_{\mathbb{R}^K} \phi(\mathbf{y}) p_1(\mathbf{y}) d\mathbf{y} = \int_{\mathbb{R}^K} p_{\mathbf{n}}(\mathbf{n}) F_1(\mathbf{n}) d\mathbf{n} = E_{\mathbf{n}}(F_1(\mathbf{n})), \tag{6}$$

where

$$F_i(\mathbf{n}) = \int_{\mathbb{R}^K} \phi(\mathbf{y}) p_i(\mathbf{y} - \mathbf{n}) d\mathbf{y}. \tag{7}$$