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A study of two-dimensional sensor placement using time-difference-of-arrival measurements

Kenneth W.K. Lui, H.C. So *

Department of Electronic Engineering, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong

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ABSTRACT

Finding the position of a radiative source based on time-difference-of-arrival (TDOA) measurements from spatially separated receivers has important applications in sonar, radar, mobile communications and sensor networks. Each TDOA defines a hyperbolic locus on which the source must lie and the position estimate can then be determined with the knowledge of the sensor array geometry. While extensive research works have been performed on algorithm development for TDOA estimation and TDOA-based localization, limited attention has been paid in sensor array geometry design. In this paper, an optimum two-dimensional sensor placement strategy is derived with the use of optimum TDOA measurements, assuming that each sensor receives a white signal source in the presence of additive white noise. The minimum achievable Cramér–Rao lower bound is also produced.

1. Introduction

Passive source localization using measurements from an array of spatially separated sensors is an important problem in radar, sonar, mobile communications and wireless sensor networks. The time-difference-of-arrival (TDOA) method is a popular strategy for source localization and it usually proceeds in a two-step fashion as follows. TDOA measurements of the source signal received at the sensor array are first obtained. In the second step, the TDOA information is utilized to construct a set of hyperbolic equations that are highly nonlinear, from which the source position can be determined with the knowledge of the sensor array geometry.

Although extensive research has been performed in TDOA estimation [1,2] as well as TDOA-based localization [3–6], most of them do not consider the impact of the sensor array geometry on the localization accuracy. Nevertheless, Yang et al. [7–9] have recently pioneered the theoretical study for sensor array placement strategies. In [7], the properties of the Cramér–Rao lower bound (CRLB) for TDOA-based positioning [3,4] and optimum sensor arrays are derived. It is proved that for two-dimensional (2D) source localization, uniform angular arrays (UAAs) and its superpositions can attain the minimum CRLB. Works in [8] and [9] are extensions to [7]. The former studies the performance loss for UAAs with reduced angular apertures while the latter develops the relationship between several sensor placement schemes. In spite of the elegant and rigorous mathematical treatment, it is invalid to assume uncorrelated TDOA measurements as they should be correlated [10]. In this paper, we will study the array geometry based on the correlated TDOA estimates, which are optimally computed using the sensor outputs where each of them receives an uncorrelated signal source in the presence of additive white Gaussian noise.

The rest of the paper is organized as follows. In Section 2, we first develop the Fisher information matrices (FIMs) for the TDOA and 2D position estimates in the case of white signal source and noise. By minimizing the trace of the CRLB for positioning, it is shown that UAAs and its superpositions correspond to an optimum sensor placement and this is in

* Corresponding author.
E-mail address: hcso@ee.cityu.edu.hk (H.C. So).

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agreement with [7]. The minimum achievable CRLB for positioning is also derived. Numerical examples are provided in Section 3 to validate our research findings. Finally, conclusions are drawn in Section 4.

2. Development of optimum sensor placement

Suppose there are $L \geqslant 3$ sensors and the signal received at the lth sensor is modeled as

$$z_l(n) = s(n - D_l) + q_l(n), \quad l = 1, 2, \ldots, L, \; n = 0, 1, \ldots, N-1, \tag{1}$$

where $s(n)$ is the white source signal, $q_l(n)$ and D_l are the additive white Gaussian noise and signal propagation delay, respectively, at the lth sensor, and N is the number of samples available at each sensor. Without loss of generality, we assign the first sensor as the reference and define the TDOA parameter vector as $\mathbf{d} = [d_{21}, d_{31}, \ldots d_{L1}]^T$ where T denotes the transpose operator and $d_{i1} = D_i - D_1$, $i = 2, 3, \ldots, L$. The FIM for $\mathbf{d}$, denoted by $\mathbf{F}(\mathbf{d})$, is [10]:

$$\mathbf{F}(\mathbf{d}) = \frac{N}{2\pi} \int_{-\pi}^{\pi} \omega^2 \frac{S^2(\omega)}{1 + \sum_{l=1}^{L} S(\omega)/Q_l(\omega)} \left[\mathrm{tr}\left(\mathbf{Q}^{-1}(\omega)\right)\mathbf{Q}_p^{-1}(\omega) - \mathbf{Q}_p^{-1}(\omega)\mathbf{1}_{L-1}\mathbf{1}_{L-1}^T\mathbf{Q}_p^{-1}(\omega)\right] d\omega, \tag{2}$$

where tr is the trace operator, $\mathbf{1}_i$ is the $i \times 1$ vector with all elements 1, $S(\omega)$ and $Q_l(\omega)$ represent the power spectra of $s(n)$ and $q_l(n)$, respectively, while $\mathbf{Q}(\omega) = \mathrm{diag}(Q_1(\omega), Q_2(\omega), \ldots, Q_L(\omega))$ and $\mathbf{Q}_p(\omega) = \mathrm{diag}(Q_2(\omega), Q_3(\omega), \ldots, Q_L(\omega))$ where $\mathrm{diag}(a_1, a_2, \ldots, a_n)$ is a diagonal matrix whose diagonal entries are $a_1, a_2, \ldots, a_n$. It is clear from (2) that the optimum TDOA estimates are correlated. Assuming that $s(n)$ and $\{q_l(n)\}$ are uncorrelated white Gaussian processes with variances σ_s^2 and σ_q^2, respectively, we have $S(\omega) = \sigma_s^2$, $Q_l(\omega) = \sigma_q^2$, $l = 1, 2, \ldots, L$, $\mathbf{Q}(\omega) = \sigma_q^2 \mathbf{I}_L$ and $\mathbf{Q}_p(\omega) = \sigma_q^2 \mathbf{I}_{L-1}$ where $\mathbf{I}_i$ represents the $i \times i$ identity matrix. Under the white signal and noise assumption, (2) can be simplified to [11]:

$$\mathbf{F}(\mathbf{d}) = \frac{\pi^2 N \Lambda^2 (L\mathbf{I}_{L-1} - \mathbf{1}_{L-1}\mathbf{1}_{L-1}^T)}{3(1 + L\Lambda)}, \tag{3}$$

where $\Lambda = \sigma_s^2/\sigma_q^2$ is the signal-to-noise ratio (SNR). The covariance matrix of the optimum estimate for $\mathbf{d}$ when $N \to \infty$ is equal to the inverse of $\mathbf{F}(\mathbf{d})$, which has the form of [11]:

$$\mathbf{F}^{-1}(\mathbf{d}) = \frac{3(1 + L\Lambda)}{\pi^2 N \Lambda^2} \left[\mathbf{I}_{L-1} + \mathbf{1}_{L-1}\mathbf{1}_{L-1}^T\right]. \tag{4}$$

Let the position of the source and sensors be $\mathbf{x} = [x, y]^T$ and $\mathbf{x}_l = [x_l, y_l]^T$, $l = 1, 2, \ldots, L$, respectively. The FIM for $\mathbf{x}$ using the optimum TDOA estimates, denoted by $\mathbf{F}(\mathbf{x})$, is [5]:

$$\mathbf{F}(\mathbf{x}) = \frac{\mathbf{G}\mathbf{F}(\mathbf{d})\mathbf{G}^T}{c^2}, \tag{5}$$

where

$$\mathbf{G} = [\mathbf{g}_{21}, \mathbf{g}_{32}, \ldots, \mathbf{g}_{L1}],$$

$$\mathbf{g}_{l1} = \mathbf{g}_l - \mathbf{g}_1,$$

$$\mathbf{g}_l = \begin{bmatrix} g_{x,l} \\ g_{y,l} \end{bmatrix} = \begin{bmatrix} \frac{x - x_l}{\sqrt{(x-x_l)^2 + (y-y_l)^2}} \\ \frac{y - y_l}{\sqrt{(x-x_l)^2 + (y-y_l)^2}} \end{bmatrix} = \begin{bmatrix} \cos(\theta_l) \\ \sin(\theta_l) \end{bmatrix}$$

and c and θ_l denote the known signal propagation speed and the incline angle from the source to the lth sensor, respectively. With the use of (3), (5) becomes

$$\mathbf{F}(\mathbf{x}) = \frac{\pi^2 N \Lambda^2}{3c^2(1 + L\Lambda)} \begin{bmatrix} (L-1)\sum_{l=1}^{L}\cos^2(\theta_l) - \sum_{i \neq j}^{L}\cos(\theta_i)\cos(\theta_j) & L\sum_{l=1}^{L}\cos(\theta_l)\sin(\theta_l) - \sum_{l=1}^{L}\cos(\theta_l)\sum_{l=1}^{L}\sin(\theta_l) \\ L\sum_{l=1}^{L}\cos(\theta_l)\sin(\theta_l) - \sum_{l=1}^{L}\cos(\theta_l)\sum_{l=1}^{L}\sin(\theta_l) & (L-1)\sum_{l=1}^{L}\sin^2(\theta_l) - \sum_{i \neq j}^{L}\sin(\theta_i)\sin(\theta_j) \end{bmatrix}. \tag{6}$$

The optimum sensor placement strategy is obtained by minimizing the trace of the CRLB for positioning, that is, $\mathrm{tr}(\mathbf{F}^{-1}(\mathbf{x}))$. In Appendix A, we have proved that $\mathrm{tr}(\mathbf{F}^{-1}(\mathbf{x}))$ can be expressed as $f(\boldsymbol{\theta})$ where $\boldsymbol{\theta} = [\theta_1, \theta_2, \ldots, \theta_L]^T$, and $f(\boldsymbol{\theta})$ is of the form:

$$f(\boldsymbol{\theta}) = \frac{L(L-1) - 2\sum_{i>j}^{L}\cos(\theta_i - \theta_j)}{L(L-2)\sum_{i>j}^{L}\sin^2(\theta_l - \theta_j) + 2L\sum_{l=1}^{L}\sum_{i>j}^{L}\sin(\theta_l - \theta_j)\sin(\theta_i - \theta_l)}. \tag{7}$$