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A novel algorithm for image representation using discrete spectrum of the Schrödinger operator

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ABSTRACT

This paper extends the recent signal analysis method based on the spectral analysis of the semi-classical Schrödinger operator to two dimensions. An algorithm based on the tensor product approach when writing the eigenfunctions of the semi-classical Schrödinger operator is proposed. The algorithm is described and the effect of some parameters on the convergence of this method are numerically studied. The performance of the algorithm is illustrated through some examples.

1. Introduction

In signal processing, it is more common to decompose a given signal into an appropriate set of functions which are independent of the signal itself. However, for some applications, signal-dependent functions are more suitable to highlight some specific features of the signal [14]. One approach to obtain such signal-dependent functions is to consider those computed from the eigenfunctions of operators which depend on the signal. The Semi-Classical Signal Analysis (SCSA) method belongs to the class of these techniques. It has been proposed by Laleg *et al.* in [14] and [10]. The main idea of this new method consists in considering the input signal as a potential of the semi-classical Schrödinger operator and then decomposing this potential using the squared L^2-normalized eigenfunctions associated to the discrete spectrum of this operator. These functions are spatially shifted and localized. Thanks to their interesting properties [10,14,20], the SCSA method has proved its performance in several applications. For instance, interesting results have been obtained in the analysis of arterial blood pressure signals [14,16,17] and the analysis of the performance of turbo-machinery [9]. Moreover, it has been shown in [18], that the SCSA method can cope with noisy signals, making this method a potential tool for denoising, for concrete example, the filtering property of the SCSA method is currently under study through in-vivo experiments with Magnetic Resonance Spectroscopy data [15].

As described in [14], the 1D version of the SCSA method has some modeling motivations related to solitons, solution of Korteweg–de-Vries (KdV) equation which have been used to model the arterial blood pressure waves. Indeed, solving the KdV equation uses the direct and inverse scattering transforms which consist in considering the solution of the KdV as potential of the Schrödinger operator [2]. Even if the method was first designed for the arterial blood pressure signals, it has been shown that it works for a broader class of signals where the idea consists in considering the signal as a multiplication operator which perturbs the semi-classical Laplacian operator which leads to the semi-classical Schrödinger operator. The semi-classical parameter plays the role of a zoom on the signal, where decreasing this parameter helps to reconstruct the details of the signal. A possible way to interpret the method and to give some intuitions on it, is the following. The inverse problem community has well studied the inverse problem of reconstructing the potential of the Schrödinger operator from its spectral data and an analytical formula linking this potential to the spectral data has been proposed in [5]. However the formula can not be used in its form since it includes terms that depend on the continuous spectrum, which is difficult to compute numerically. In [14], it has been shown that the introduction of the semi-classical parameter in the Schrödinger operator can reduce significantly the effect of the continuous spectrum which justifies the use of the semi-classical analysis in the proposed approach and which also explains the zoom role of the semi-classical parameter.

In this paper, the SCSA method was extended to two dimensions (2D) to be used in image representation. The reconstruction formula is first derived for the 2D case thanks to some concepts from the semi-classical analysis of the Schrödinger operator [11,13]. Then an efficient algorithm is proposed where the

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two dimensional (2D) semi-classical Schrödinger operator is splited into two one dimensional (1D) operators and then the squared L^2-normalized eigenfunctions of these 1D operators are combined using a tensor product approach [8,12].

This paper is organized as follows. In Section 2, the basic properties of the SCSA are recalled. Then, in Section 3, the extension of the SCSA formula to 2D is proposed. In Section 4, an algorithm for image representation, based on the spectral problems of 1D Schrödinger operators is introduced. The analysis of some parameters and the use of this algorithm in image representation is illustrated in Section 5, followed by a comparison with algorithms from state-of-the-art image representation methods [4,23]. The last Section summarizes and discusses the obtained results.

2. Preliminary (SCSA in 1D case)

In this section, we recall the definition of the SCSA method [10, 14]. Let us consider the following 1D operator, known as the semi-classical Schrödinger operator:

$$\mathcal{H}_{1,h}(V_1)\psi = -h^2\frac{d^2\psi}{dx^2} - V_1\psi, \quad \psi \in \mathbf{H}^2(\mathbb{R}) \tag{1}$$

where $h \in \mathbb{R}_+^*$ is the semi-classical parameter [6], and V_1 is a positive real valued function belonging to $\mathcal{C}^\infty(\Omega_1)$ where $\Omega_1 =]a, b[$ is a bounded open interval. Here $\mathbf{H}^2(\mathbb{R})$ denotes the Sobolev space of order 2. The potential V_1 can be represented using the following proposition.

Proposition 2.1. *(See [10].) Let $V_1 \in \mathcal{C}^\infty(\Omega_1)$ be a positive real valued function, and $\Lambda_1 \subset \Omega_1$ is compact. Then, V_1 can be reconstructed in Λ_1 using the following expression:*

$$V_{1,h,\gamma,\lambda}(x) = -\lambda + \left(\frac{h}{L^{cl}_{1\gamma}}\sum_{k=1}^{K_h^\lambda}(\lambda-\mu_{k,h})^\gamma\,\psi_{k,h}^2(x)\right)^{\frac{2}{1+2\gamma}},$$
$$x \in \Lambda_1, \tag{2}$$

where $h \in \mathbb{R}_+^*$, $\gamma \in \mathbb{R}_+$, $\lambda \in \mathbb{R}_-$, and $L^{cl}_{1,\gamma}$ is the suitable universal semi-classical constant given by:

$$L^{cl}_{1,\gamma} = \frac{1}{2\sqrt{\pi}}\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\frac{3}{2})},$$

Γ refers to the standard Gamma function. $\mu_{k,h}$ are the negative eigenvalues of $\mathcal{H}_{1,h}(V_1)$ with $\mu_{1,h} < \cdots < \mu_{K_h^\lambda,h} < \lambda$, K_h^λ is a finite number of negative eigenvalues smaller than λ, and $\psi_{k,h}$ are the associated L^2-normalized eigenfunctions such that,

$$\mathcal{H}_{1,h}(V_1)\,\psi_{k,h} = \mu_{k,h}\psi_{k,h}, \quad k = 1,\cdots,K_h^\lambda.$$

The influence of the parameters λ, γ and h has been studied in [10] where it has been shown that the semi-classical parameter h plays a key role in this method. Indeed, when h goes to 0, the analysis of the Schrödinger operator (1) is usually related to the semi-classical analysis [6], which justifies the name Semi-Classical Signal Analysis (SCSA) for this signal analysis method [10,14].

3. Extension of the SCSA method to two-dimension

We consider the following 2D semi-classical Schrödinger operator associated to a potential V_2:

$$\mathcal{H}_{2,h}(V_2) = -h^2\Delta - V_2, \tag{3}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the 2D Laplacian operator, $h \in \mathbb{R}_+^*$ is the semi-classical parameter [6], and V_2 is a positive real valued function belonging to $\mathcal{C}^\infty(\Omega_2)$ where $\Omega_2 =]a, b[\times]c, d[$ is a bounded open set of $\mathbb{R}^2$.

Then, inspired form the semi-classical properties of the Schrödinger operator [11,13], the extension of the SCSA formula in 2D case is given by the following theorem.

Theorem 3.1. *Let V_2 be a positive real valued $\mathcal{C}^\infty$ function on Ω_2 considered as potential of the Schrödinger operator (3). Then, for any pair (Λ_2, λ) such that $\Lambda_2 \subset \Omega_2$ is compact and*

$$\begin{cases} \lambda < \inf(-V_2(a,c), -V_2(b,d)), \\ V_2(\Lambda_2) \subset]-\lambda, +\infty[, \\ \lambda \text{ is not a critical value of } -V_2, \ (\text{for more details see [11]}) \end{cases} \tag{4}$$

and, uniformly for $(x, y) \in \Lambda_2$, we have

$$V_2(x,y) = -\lambda + \lim_{h\to 0}\left(\frac{h^2}{L^{cl}_{2,\gamma}}\sum_{k=1}^{K_h^\lambda}(\lambda-\mu_{k,h})^\gamma\,\psi_{k,h}^2(x,y)\right)^{\frac{1}{1+\gamma}}, \tag{5}$$

where $\gamma \in \mathbb{R}_+^*$, $L^{cl}_{2,\gamma}$ is the suitable universal semi-classical constant given by:

$$L^{cl}_{2,\gamma} = \frac{1}{2^2\pi}\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+2)}, \tag{6}$$

Γ refers to the standard Gamma function.

Moreover, $\mu_{k,h}$ and $\psi_{k,h}$ denote the negative eigenvalues with $\mu_{1,h} < \cdots < \mu_{K_h^\lambda,h} < \lambda$, K_h^λ is a finite number of negative eigenvalues smaller than λ, and associated L^2-normalized eigenfunctions of the operator $\mathcal{H}_{2,h}(V_2)$ such that:

$$\mathcal{H}_{2,h}(V_2)\,\psi_{k,h} = \mu_{k,h}\,\psi_{k,h}, \quad k = 1,\cdots,K_h^\lambda \tag{7}$$

The proof of this result is obtained using the generalization of Theorem 4.1, proposed by Helffer and Laleg in [10], to 2D case, which uses an extension of Karadzhov's theorem on the spectral function [13], together with the connection of the Riesz means with Lieb-Thirring conjecture proposed by Helffer and Robert in [11]. Details of the proof are provided in Appendix A.

4. A novel algorithm for image representation

In image processing, for some geometrical and topological reasons, it is common and more practical to consider a separation of variables approach to extend the 1D transforms to 2D [8,12]. This is the case for example with the 2D Fourier transform, which can be written using the tensor product of the 1D complex exponential [19] or more recently the Ridgelet transform [7] based on the tensor product of 1D wavelet transform. The separation of variables principle allows the design of efficient and fast algorithms where the representation of the image is done row by row and column by column.

The reconstruction of an image using formula (5) requires the computation of eigenvalues and eigenfunctions which is known to be complex, especially in 2D. Therefore for the sake of simplicity, we propose, in this section, to use the separation of variables principle by splitting the 2D operator into two 1D operators and to solve the eigenvalues problem for these 1D operators.

4.1. Principle

Let us define, for $(x_0, y_0) \in \Lambda_2$ the following 1D operators, the same value of h is taken for both operators.