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An unconstrained optimization approach to empirical mode decomposition

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ABSTRACT

Empirical mode decomposition (EMD) is an adaptive (data-driven) method to decompose non-linear and non-stationary signals into AM-FM components. Despite its well-known usefulness, one of the major EMD drawbacks is its lack of mathematical foundation, being defined as an algorithm output. In this paper we present an alternative formulation for the EMD method, based on unconstrained optimization. Unlike previous optimization-based efforts, our approach is simple, with an analytic solution, and its algorithm can be easily implemented. By making no explicit use of envelopes to find the local mean, possible inherent problems of the original EMD formulation (such as the under- and overshoot) are avoided. Classical EMD experiments with artificial signals overlapped in both time and frequency are revisited, and comparisons with other optimization-based approaches to EMD are made, showing advantages for our proposal both in recovering known components and computational times. A voice signal is decomposed by our method evidencing some advantages in comparison with traditional EMD and noise-assisted versions. The new method here introduced catches most flavors of the original EMD but with a more solid mathematical framework, which could lead to explore analytical properties of this technique.

1. Introduction

Empirical Mode Decomposition (EMD) [1] is an adaptive method introduced to analyze non-linear and non-stationary signals. It consists in a local and fully data-driven separation of a signal in fast and slow oscillations. At the end of the decomposition, the original signal can be expressed as a sum of amplitude and frequency modulated (AM–FM) functions called *intrinsic mode functions* (IMFs) plus a final trend either monotonic or constant. However, EMD experiences some problems, such as the presence of oscillations of very disparate amplitude in a mode, or the presence of very similar oscillations in different modes, named as *mode mixing* (an interesting strategy to alleviate noise-related mode mixing can be found in [2]). Besides this issue, one of its major drawbacks is the lack of mathematical framework, being defined as an algorithm output.

Several efforts have been made in order to provide some mathematical foundations for EMD. Deléchelle et al. [3] estimated the envelopes by solving a parabolic differential equation. Xu et al. [4] modified the envelope definition to obtain one with a simple analytical expression, by which the variations of the extrema in the iterative procedure are investigated in detail to reveal the nature of the sifting process. Hawley et al. [5] replaced the cubic spline interpolations for trigonometric interpolations when estimating the envelopes. Thanks to that, they derive some interesting properties and convergence guarantees, although the results significantly differ from those of classical EMD. Daubechies et al. [6,7] compared EMD with wavelet theory by using a special case of reassignment called *synchrosqueezing*.

A different approach, based on optimization theory, has recently aroused the interest of the EMD scientific community. B. Huang and Kunoth [8] replaced the explicit interpolation through extrema for solving an optimization problem to estimate the envelopes. However, they keep an envelope-related approach. No explicit envelope is estimated by Oberlin et al. in [9]. They search for a local mean in a specific B-spline space subject to some symmetry constraints on the amplitude of the modes. In a similar way, Pustelnik et al. [10,11] minimized the difference between a signal and its local mean plus mode subject to smoothness, symmetry

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and quasiorthogonality requirements, based on a multicomponent variational analysis.

In this paper, we propose a new approach for EMD based on optimization, with the goal of mimic EMD. Unlike those above mentioned, here we present an unconstrained optimization problem, with a unique analytic solution. This results in the following benefits:

- This approach may help to better understand some properties of EMD.
- The explicit computation of envelopes to find the local mean is not needed, in contrast to algorithm-based EMD.
- The proposed method provides an analytical and easily implemented closed solution, unlike previous optimization-based efforts that need iterative algorithms to solve the optimization problem.
- The use of explicit spline interpolations is avoided.
- The number of parameters to be tuned has been reduced to only one, in contrast to other optimization-based proposals where several parameters are needed.
- The computational cost is similar to the cost of EMD. On the contrary, other optimization-based methods are tens of times slower than EMD.

The paper is organized as follows. We recall the basic principles of EMD in Section 2. In Section 3 we present our new unconstrained optimization approach to EMD. Experiments and results with both artificial and real signals are introduced and discussed in Section 5. Conclusions are presented in Section 6.

2. Empirical mode decomposition

The main idea on EMD is to iteratively subtract the local mean from a signal (or residue) to obtain the zero local mean AM–FM components called intrinsic mode functions or simply modes. From this perspective, the slow oscillation is considered the local mean (trend) while the mode is the fast one. If x is the signal to be decomposed, the EMD algorithm can be summarized as follows [1]:

1. Set $k = 0$ and find all extrema of $r_0 = x$.
2. Interpolate between minima (maxima) of r_k to obtain the lower (upper) envelope e_{min} (e_{max}).
3. Compute the mean envelope $m = (e_{min} + e_{max})/2$.
4. Compute the IMF candidate $d_{k+1} = r_k - m$.
5. Is d_{k+1} an IMF?
   - Yes. Save d_{k+1}, compute the residue $r_{k+1} = x - \sum_{i=1}^{k+1} d_i$, $k = k + 1$, and treat r_k as input data in step 2.
   - No. Treat d_{k+1} as input data in step 2.
6. Continue until the final residue r_K satisfies some predefined stopping criterion.

At the end, the signal x can be expressed as

$$x = \sum_{i=1}^{K} d_k + r_K, \tag{1}$$

where each mode d_k admits well-behaved Hilbert transforms. The refinement process carried out to ensure that the mode d_k is actually an IMF is the so-called *sifting process*. Further details can be found in [1].

The symmetry of the modes' envelopes resides on the IMF definition. To be considered an IMF, a function must fulfill two conditions: (i) the number of extrema and zero crossings are equal or differ at most by one; and (ii) the mean between the upper and lower envelope is zero for all the signal duration.

Some of the main characteristics of the EMD are its multiscale and local nature. The local scale is defined as the interval between successive extrema. The number of extrema of the modes decreases as k increases. Although one may give "spectral" interpretation of the modes, it must be emphasized that this applies only locally. The automatic selection of the local highest frequency content cannot be achieved by a predetermined subband filtering; it rather corresponds to an adaptive (data-dependent) time-variant filtering [12]. When decomposing fractional Gaussian noise (fGn), EMD acts on average as a dyadic filter bank [12,13].

3. EMD as an unconstrained non-linear convex optimization problem

Notice that, in the original EMD algorithm, the local mean is defined as the mean of the envelopes, which are obtained by interpolating through local extrema (usually with cubic splines). Therefore, from the second mode onwards, all of them are sums of splines. We must get rid of the envelopes, so we propose here a different approach to obtain the local mean. Previous efforts have focused their attention on the smoothness of the local mean [14] (even restricting their search to a spline subspace [9]). In those approaches, the IMF-likeness of the modes is not considered on the objective function but in the form of inequality constraints, where the corresponding bounds have to be set. However, the IMF conditions are the heart of EMD and the sifting process is carried out until the mode is close enough to an IMF, so the original signal is the sum of IMFs plus a final trend. For this reason, our proposal focuses on the IMF-likeness of the modes.

Let us return to the IMF definition in Section 2. It is clear that condition (ii) cannot be satisfied without fulfilling condition (i), so it is enough to pursuit condition (ii). We consider this issue in a similar fashion to Oberlin et al. [9] and Pustelnik et al. [10]. Let $t_k[l]$, $1 \le l \le L$, with L the number of local extrema, be the locations of the local extrema of the signal (or residue) under study. If we consider these points as estimations of the local extrema locations of the k-th mode (d_k), for $2 \le l \le L - 1$, we can define the inner product

$$p^k_{t_k[l]}\, d_k = d_k(t_k[l]) + \frac{d_k(t_k[l+1])\Delta_l^- + d_k(t_k[l-1])\Delta_l^+}{\Delta_l^+ + \Delta_l^-}, \tag{2}$$

where $\Delta_l^+ = t_k[l+1] - t_k[l]$, $\Delta_l^- = t_k[l] - t_k[l-1]$, d_k is a column vector and $p^k_{t_k[l]}$ is the $t_k[l]$-th row of a matrix P^k. Then, the only non-zero elements of the $t_k[l]$-th row of P^k are

$$P^k(t_k[l], t_k[l]) = 1, \tag{3a}$$

$$P^k(t_k[l], t_k[l-1]) = \frac{\Delta_l^+}{\Delta_l^+ + \Delta_l^-}, \tag{3b}$$

$$P^k(t_k[l], t_k[l+1]) = \frac{\Delta_l^-}{\Delta_l^+ + \Delta_l^-}. \tag{3c}$$

It should be emphasized the fact that matrix P^k has as many rows as the length of d_k. Rows not involved in (2) are zeros, because there are no local extrema at that positions. The goal of (2) is to compare the signal (or residue) at each extrema with the corresponding linear interpolation between its two adjacent extrema. Smaller values of (2) would mean that d_k locally (around $t_k[l]$) better satisfies the IMF condition (ii). The minimization of $||P^k d_k||^2$ would contribute, at least globally, to the fulfillment of the IMF conditions. As it was pointed out by Pustelnik et al., matrix P is a *"...linear operator which models the penalization imposed on d at each location* $t_k[l]$*"* [10]. It must be noticed that, in this approach, the IMF conditions are not evaluated over the whole time span of the signal but only on its local extrema.