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Separation of variables in one case of motion of a gyrostat acted upon by gravity and magnetic fields

M.P. Kharlamov [a,*], H.M. Yehia [b]

[a] Math. Simul. Department, Russian Academy of National Economy and Public Administration, Volgograd Branch, Volgograd 400131, Russia
[b] Math. Department, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

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ABSTRACT

A version of the integrable problem of motion of a dynamically symmetric gyrostat about a fixed point similar to the Kowalevski top, while acted upon by a combination of uniform gravity and magnetic fields is considered. This version is reduced, in general, to hyper-elliptic quadratures. The special case when the gyrostatic momentum is absent is solved in terms of elliptic functions of time.

1. Introduction

The problem of motion of a heavy rigid body about a fixed point has a long history beginning with Euler's work [1]. For most of this history, the main concern of authors was to isolate cases, when the general solution of the equations of motion can be expressed explicitly in terms of functions of time, or, at least, can be reduced to quadratures. This recipe has succeeded in two cases: Euler's case of a body moving by inertia and Lagrange's case of a symmetrical top [2]. Separation of variables in Euler's case was found by Euler himself, but the solution was expressed by Jacobi in terms of his newly invented elliptic functions. Lagrange reduced the case of axisymmetric top to separation of variables involving elliptic integrals. Explicit expression of the solution in terms of time was initiated by Jacobi and can be found with some variations in [3–5].

The following historical turn in rigid body dynamics came in the opposite direction, from the study of the nature of solutions of the equations of motion. Kowalevski [6] isolated the possible cases which share with Euler's and Lagrange's cases

* Corresponding author.
E-mail addresses: mikeh@inbox.ru (M.P. Kharlamov), hyehia@mans.edu.eg (H.M. Yehia).
URL: http://vlgr.ranepa.ru/pp/hmp

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the property of having their solutions as meromorphic functions of time. It turned out that only one more case satisfies this criterion. That case became known as Kowalevski's. Kowalevski obtained the complementary integral for that case as a quartic polynomial in velocities. She also integrated the equations of motion in terms of hyperelliptic functions of time. Her solution was simplified by Kötter [7] and reconsidered by a series of authors. For a detailed history, see e.g. [8].

When a uniformly rotating rotor has its axis of symmetry fixed in the rigid body, the resulting system is known as a gyrostat. For this system, the generalization of Lagrange's case and its solution was straightforward. Euler's case was generalized by Zhukovsky [9]. The corresponding solution was shown by Volterra to be expressible in terms of sigma functions of Weierstrass [10]. In [11], Wittenburg pointed out another solution in terms of elliptic functions of time. Detailed presentation of the history of general and particular solutions for a heavy gyrostat can be found in [12].

A century after the discovery of Kowalevski's case, its generalization to the problem of gyrostat has been found in a different context. The problem of motion of a gyrostat similar to the Kowalevski top and acted upon by two skew uniform fields (gravity and magnetic) was considered in [13]. A general first integral quartic in velocities and generalizing the Kowalevski integral was found for this generic problem. As, in the general case, the two force fields problem does not admit a symmetry group, the additional cyclic integral does not exist. It turned out that such integral still exists in two special cases [13]. The first case generalizes Kowalevski's case of one field to the gyrostat motion in one field. The second case does not contain the classical case of Kowalevski, since the intensities of the two fields are proportional and can vanish only simultaneously. In the last case the cyclic variable is a complementary angle to the sum (or difference) of the two angles of precession and proper rotation.

In the present paper we accomplish separation of variables for a version of the last case, corresponding to a special value of the cyclic constant proportional to the gyrostatic moment and singled out by the condition that the reduced system becomes time-reversible. We give two algebraic separations of variables. In the first one, the cyclic constant is supposed nonzero and the variables of separation are to be determined as functions of time by solving hyperelliptic Abel–Jacobi equations. This result is based on the analogy established in [14] (see also [15] for further generalizations) of a special class of problems of the gyrostat motion in two fields with the problems of the gyrostat motion in axially symmetric field with zero momentum constant. Thus, the first separation given below corresponds to the algebraic separation [16,17] found by the method proposed in [18,19] for the Goryachev case. This separation is not applicable if the gyrostatic moment in the initial problem is zero. Nevertheless, as it is shown in [14], the equivalent problem of the rigid body motion in an axisymmetric field is the integrable case of Chaplygin [20]. Therefore we give the second separation which transfers the elliptic separation found by Chaplygin to the two-fields problem.

The two types of separation of variables accomplished here make it easy to apply the algorithm of finding the admissible regions for the integral constants and to establish the rough phase topology of the system [21,22]. Moreover, the recent results for separated systems [23] give a method to calculate the exact topological invariants of singular points and all regular iso-energy levels. This will give the complete topological analysis of the problem which will be different (as far as special types of motions are concerned) from the corresponding Goryachev and Chaplygin cases [17,24] since the analogy of these problems with the motion of a gyrostat in two fields does not give a global diffeomorphism of the corresponding phase spaces.

It will also be interesting to analyze all special cases when the trajectories in reduced systems become periodic. For the Goryachev case the corresponding quadratures are found in [17]. In our problem such quadratures can lead to explicit calculation of the orientation matrix and therefore provide the analytical basis for the geometric interpretation of periodic and two-frequency motions of the considered gyrostat in two fields.

2. Equations and integrals

The equation of the motion of a gyrostat acted upon by two homogeneous fields in the general case can be written in the Euler – Poisson form

$$\dot{\mathbf{M}} = \mathbf{M} \times \boldsymbol{\omega} + \mathbf{c}_1 \times \boldsymbol{\alpha} + \mathbf{c}_2 \times \boldsymbol{\beta}, \qquad \dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \boldsymbol{\omega}. \tag{1}$$

Here $\boldsymbol{\omega}$ is the angular velocity, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the characteristic vectors of the force fields (say, the vectors of the gravity force and of the magnetic field strength), $\mathbf{c}_1, \mathbf{c}_2$ are the vectors pointing from the fixed point O to the centers of the fields application. All objects are referred to some moving axes. The kinetic momentum vector $\mathbf{M}$ is connected with the angular velocity by the relation

$$\mathbf{M} = \boldsymbol{\omega}\mathbf{I} + \boldsymbol{\lambda},$$

where $\mathbf{I}$ and $\boldsymbol{\lambda}$ are the inertia tensor at O and the gyrostatic momentum vector. Both $\mathbf{I}$ and $\boldsymbol{\lambda}$ are constant in the moving frame. We consider the components of all vectors as rows, thus obtaining the unusual order of the objects in the above expression for $\mathbf{M}$.

It is known [8] that without changing the plane $O\mathbf{c}_1\mathbf{c}_2$ in the body, one can make the pair of the vectors $\mathbf{c}_1, \mathbf{c}_2$ to be orthonormal. Let us choose the moving frame $O\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ of the principal axes of the inertia tensor. Suppose that the gyrostat is dynamically symmetric $\mathbf{e}_1\mathbf{I}\cdot\mathbf{e}_1 = \mathbf{e}_2\mathbf{I}\cdot\mathbf{e}_2$, $\boldsymbol{\lambda} = \{0, 0, \lambda\}$ and the centers of the fields application lie in the equatorial plane $\mathbf{c}_1\cdot\mathbf{e}_3 = 0$, $\mathbf{c}_2\cdot\mathbf{e}_3 = 0$. In this case (see [25,26]) by some linear change of variables one can make the immovable in space vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}$ to be mutually orthogonal. Then, after the pair $\mathbf{c}_1, \mathbf{c}_2$ is made orthonormal, the modules of the vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}$ contain all scalar information on the interaction of the gyrostat with the fields (e.g. for the gravity field, the module of the corresponding vector is equal to the product of the gyrostat weight and the distance from the mass center to the fixed point). Therefore, we call $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ the intensities of the force fields. In the case of dynamic symmetry any orthonormal pair in the equatorial plane becomes principal for the inertia tensor, so we take $\mathbf{e}_1 = \mathbf{c}_1$, $\mathbf{e}_2 = \mathbf{c}_2$.