Mechanical Systems and Signal Processing 79 (2016) 182–191

Contents lists available at ScienceDirect

Mechanical Systems and Signal Processing

journal homepage: www.elsevier.com/locate/ymssp

# On some approximations of the resultant contact forces and their applications in rigid body dynamics

Grzegorz Kudra*, Michał Szewc, Igor Wojtunik, Jan Awrejcewicz

*Lodz University of Technology, Department of Automation, Biomechanics and Mechatronics, Stefanowski St. 1/15, 90-924 Lodz, Poland*

ARTICLE INFO

*Article history:*
Received 15 February 2016
Accepted 23 February 2016
Available online 8 March 2016

*Keywords:*
Friction modeling
Coulomb–Contensou model
Plane contact
Rolling resistance

ABSTRACT

The work presents the possible applications and effectiveness of certain class of models of the resultant friction force and rolling resistance. The friction models are based on the integral model constructed under assumption of fully developed sliding on the plane contact area of general shape and any pressure distribution. Then the integral model of friction force and moment are approximated based on Padé approximants and their generalizations. These models are expected to be computationally effective in numerical simulations of rigid bodies with frictional contacts, such like billiard balls, Thompson top, the wobble stone and many others. In the present work two different examples of application of the developed contact models are presented and tested: a) a billiard ball rolling and sliding on the plane horizontal table; b) a full ellipsoid of revolution in contact with plane and horizontal base.



## 1. Introduction

There are examples of mechanical systems with frictional contacts, where one-dimensional friction model is insufficient to obtain correct simulation results. One can solve then the full contact problem numerically by using space discretization. It requires, however, the use of such numerical methods like finite element method or, at least, the integration over the contact area, and leads to great increase of computational cost. From the point of view of fast and realistic numerical simulations of some class of rigid bodies, it is important to develop special approximate models of resultant friction force and moment.

In 1962 Contensou wrote a work [1], where he presented integral model of the resultant friction force, assuming fully developed sliding and classical Coulomb law of friction valid on each element of plane circular contact area, where Hertzian contact pressure distribution is applied. Some group of researchers proposed a model of approximate friction force and moment based on an ellipsoid approximating the surface bounding the set of admissible tangential loading of the contact [2]. It is relatively good model for determination of limit friction forces, for which the relative motion does not appear. However, the use of this model for determination of the relation between the sliding direction and the resultant friction components, leads to much larger errors. In order to determine the sliding direction corresponding to each point of the limit surface, they used the property of friction based on that the friction force direction maximizes the work on a given displacement. It results in convex shape of the admissible tangential loading of the contact. The normal direction to the limit surface determine then the direction of possible sliding [3].

* Corresponding author. Tel.: +48 42 6312339.
*E-mail address:* grzegorz.kudra@p.lodz.pl (G. Kudra).

http://dx.doi.org/10.1016/j.ymssp.2016.02.036
0888-3270/

Then some researchers developed the results of Contensou giving exact analytical functions defining the resultant friction force and torque on circular contact area with the Hertzian stress distribution, as well as proposing their special approximations based on Padé approximants of the order (1,1), more suitable and convenient for numerical simulations [4]. In the work [5] one proposed the Padé approximations of the order (2,2), more adequate for qualitative bifurcational analysis. In the next work [6] there is presented a model of the resultant friction force and moment additionally coupled with rolling resistance. In this model the initial Hertzian contact pressure distribution was distorted in a special way in order to generate the rolling friction. In the work [7] there is proposed a piece-wise linear approximation of the integral model of friction components on the elliptic contact area, assuming the contact pressure distribution according the Hertz theory. Another approach is represented in the work [8], where the coupled model of friction force and moment on a circularly symmetric contact pressure is obtained by the use of Taylors' expansions of the pseudopotentials of velocity. In the next step the friction models were applied in the Thompson top modeling and simulations. The authors of this work proposed a family of approximant models of friction forces, which can be seen as a kind of generalization of the earlier models based on Padé approximants [9]. They applied them in modeling and simulations of the Celtic stone, assuming elliptical frictional contact with the presence of rolling resistance [10].

In the present work we outline the previously obtained results concerning approximate modeling of the resultant friction forces on elliptic contact area along with the rolling resistance (Section 2) and then we present two examples of their application: dynamics of a billiard ball rolling and sliding over plane surface (Section 3) and a full ellipsoid of revolution in contact with plane and horizontal base (Section 4).

## 2. Modeling of the contact forces

### 2.1. Integral model of resultant friction forces for general plane contact

We consider a dimensionless form of a plane contact area $F$ presented in Fig. 1a, with the Cartesian coordinate system $Axyz$, where the axes $x$ and $y$ lie in the plane of the contact and the dimensionless length is defined as the quotient of the actual length and a characteristic dimension $\hat{a}$ of the contact. It is assumed that the sliding is fully developed and the relative motion of the two bodies can be treated as a plane motion of rigid bodies. This motion is then a function of the following quantities: $\mathbf{v}_s=\hat{v}_s/\hat{a}=v_{sx}\mathbf{e}_x+v_{sy}\mathbf{e}_y$ – the dimensionless linear sliding velocity at the pole $A$, $\boldsymbol{\omega}_s=\hat{\boldsymbol{\omega}}_s=\omega_s\mathbf{e}_z$ – the dimensionless angular sliding velocity ($\hat{\mathbf{v}}_s$ and $\hat{\boldsymbol{\omega}}_s$ denote the corresponding real counterparts), where $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ are the unit vectors of the corresponding axes.

Additionally, the Coulomb friction law is applied on each element $dF$: $d\mathbf{T}_s=-\sigma(x,y)dF\,\mathbf{v}_P/\|\mathbf{v}_P\|=d\hat{\mathbf{T}}_s/(\mu\hat{N})$, where $d\mathbf{T}_s$ and $d\hat{\mathbf{T}}_s$ are, correspondingly, the non-dimensional and real elementary friction forces, $\sigma(x,y)=\hat{\sigma}(x,y)\,\hat{a}^2/\hat{N}$ – dimensionless contact pressure distribution (where $\hat{\sigma}(x,y)$ is its real counterpart), $\mathbf{v}_P$ – local dimensionless velocity of sliding, $\hat{N}$ – the normal component of the total real interaction between the contacting bodies, $\mu$ – dry friction coefficient. The moment of the elementary friction force about the nominal center $A$ of the contact reads $d\mathbf{M}_s=\boldsymbol{\rho}\times d\mathbf{T}_s=d\hat{\mathbf{M}}_s/(\hat{a}\mu\hat{N})$, where $d\hat{\mathbf{M}}_s$ is the real counterpart of $d\mathbf{M}_s$.

The elementary friction forces $d\mathbf{T}_s$ and moments $d\mathbf{M}_s$ can be summed up, giving the total friction force $\mathbf{T}_s=-T_{sx}\mathbf{e}_x-T_{sy}\mathbf{e}_y$ acting in the point $A$ and moment $\mathbf{M}_s=-M_s\mathbf{e}_z$, where

$$T_{sx}=\iint_F\frac{\sigma(x,y)(v_{sx}-\omega_s y)}{\sqrt{(v_{sx}-\omega_s y)^2+\left(v_{sy}+\omega_s x\right)^2}}dxdy,\quad T_{sy}=\iint_F\frac{\sigma(x,y)\left(v_{sy}+\omega_s x\right)}{\sqrt{(v_{sx}-\omega_s y)^2+\left(v_{sy}+\omega_s x\right)^2}}dxdy,$$

$$M_s=\iint_F\sigma(x,y)\frac{\omega_s\left(x^2+y^2\right)+v_{sy}x-v_{sx}y}{\sqrt{(v_{sx}-\omega_s y)^2+\left(v_{sy}+\omega_s x\right)^2}}dxdy. \tag{1}$$

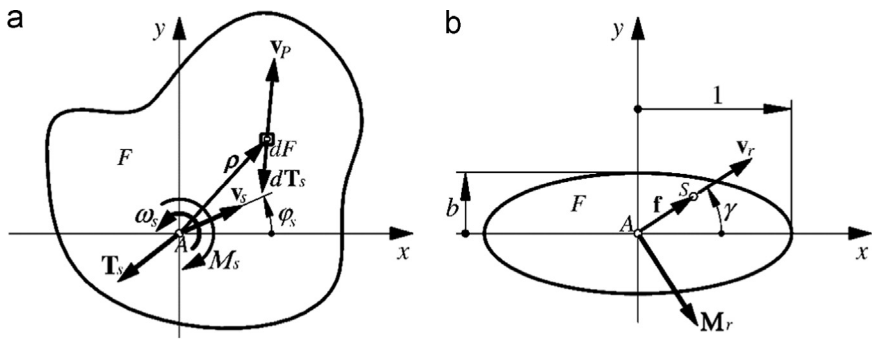


**Fig. 1.** The contact area: the general case (a) and a special case of elliptical shape (b).