Digital Signal Processing 38 (2015) 32–42

Contents lists available at ScienceDirect

Digital Signal Processing

www.elsevier.com/locate/dsp

The theoretical analysis for an iterative envelope algorithm ☆

Lijun Yang [a,b], Zhihua Yang [c], Lihua Yang [b,*]

[a] School of Mathematics and Statistics, Henan University, China
[b] Guangdong Province Key Laboratory of Computational Science, School of Mathematics and Computational Science, Sun Yat-sen University, China
[c] Information Science School, Guangdong University of Finance & Economics, China

ARTICLE INFO

Article history:
Available online 30 December 2014

Keywords:
Envelope
Undershoot
Monotone piecewise cubic interpolation (MPCI)
Convergence rate

ABSTRACT

Cubic spline interpolating the local maximal/minimal points is often employed to calculate the envelopes of a signal approximately. However, the undershoots occur frequently in the cubic spline envelopes. To improve them, in our previous paper we proposed a new envelope algorithm, which is an iterative process by using the Monotone Piecewise Cubic Interpolation. Experiments show very satisfying results. But the theoretical analysis on why and how it works well was not given there. This paper establishes the theoretical foundation for the algorithm. We will study the structure of undershoots, prove rigorously that the algorithm converges to an envelope without undershoots with exponential rate of convergence, which can be used to determine the number of iterations needed in the algorithm for a good envelope in applications.

1. Introduction

Envelope is one of the most fundamental concepts in signal processing. It plays an important role in the instantaneous frequency estimation, demodulation of mono-component signals, adaptive decomposition and so on. Intuitively speaking, for a given signal, *the envelope should be smooth and contain as little oscillation as possible; it should be situated above (below) the signal except for the points where the envelope and the signal are tangent, and wrap the signal as tightly as possible.* Although the physical intuition of envelope looks simple, a rigorous mathematical definition remains an open issue. Some practical models have been developed and used widely in signal analysis and processing [1,7,11].

The earliest mathematical model for envelope can be traced back to the analytic signal (AS) method, which was first proposed by D. Gabor to estimate the instantaneous frequency and amplitude of a signal [3]. Given a real-valued signal $x(t)$, its Hilbert transform is defined as the following Cauchy principal value integral

$$Hx(t) = \text{p.v.} \int_{\mathbb{R}} \frac{x(t-\tau)}{\tau} d\tau. \tag{1}$$

By taking $x(t)$ as the real part and $Hx(t)$ as the imaginary part we obtain the analytic signal $z(t) = x(t) + iHx(t) = \rho(t)e^{i\theta(t)}$, where

$$\rho(t) = \sqrt{x^2(t) + \left[Hx(t)\right]^2} \tag{2}$$

is defined as the analytic envelope of the signal $x(t)$. This technique for computing the envelope is called the AS method and the resulting envelope $\rho(t)$ is called the Analytic Signal Envelope (ASE). This model was proved to be the unique reasonable analytic model for signal demodulation under some mild conditions by Vakman [14,15]. Recently, J.F. Huang and L.H. Yang gave a solid mathematical foundation for this result by extending the Hilbert transform to a larger space $L^p_w(\mathbb{R})$ [5,6]. Even though this model is complete from the point of mathematical view, however, from the point of physical view, it often produces ridiculous results for practical signals [4,16]. In 1998, N.E. Huang and his coworkers computed the envelope by interpolating the maxima/minima of the input signal with the cubic spline [7], which we denote by CSE. It is shown that the CSE coincides with the physical envelope of the signal much better than the ASE in most cases [16]. But this model has an inherent drawback: the undershoots often occur near the local extreme points of the signal, which contradicts the physical requirement that the upper/lower envelope situates above/below the signal. To improve the CSE model, we proposed a new envelope algorithm by modifying the envelope iteratively using the monotonic piecewise cubic interpolation (MPCI) in our recently published paper [16], which is denoted by IMCE. Experiments show that the IMCEs can eliminate the undershoots completely and meanwhile keep the smoothness property. Fig. 1 is an

☆ This work is supported by National Natural Science Foundation of China (Nos. 11426087, 11371017, 11431015), the Research Fund for the Doctoral Program of Higher Education of China (No. 20130171110016) and the Research Foundation of Henan University (No. 2013YBZR016).

* Corresponding author. Fax: +86 20 84037978.
E-mail address: mcsylh@mail.sysu.edu.cn (L. Yang).

http://dx.doi.org/10.1016/j.dsp.2014.12.006
1051-2004/© 2014 Elsevier Inc. All rights reserved.

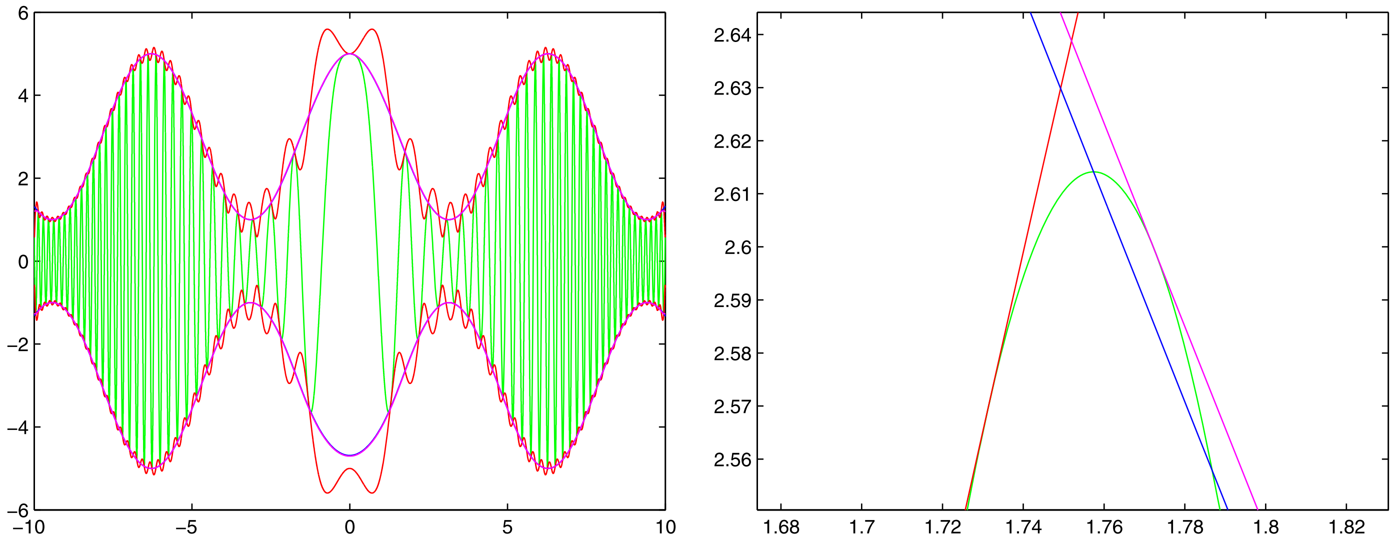

Fig. 1. The comparison of the envelope methods for the signal $x(t) = (3 + 2\cos(t))\cos(2t^2)$. Left: The original signal (green), ASE (red), CSE (blue) and IMCE (purple); Right: The drawing of partial enlargement. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

example which compares these methods and shows that the IMCE has completely eliminated the undershoots. Since the CSE plays a key role in the Empirical Mode Decomposition and the AM/FM demodulation of Intrinsic Mode Functions [7,8], evident gain can be obtained when we substitute the improved envelope IMCE for the CSE [16]. Lots of experiments show that, for general signals, all the undershoots can be eliminated after a few times of iteration procedures, usually 2–3 times. However, the theoretical proof was not provided there. In this paper, we prove that iterative procedure converges exponentially to an envelope without undershoots.

The study on envelope is an active field in time-frequency and non-stationary signal processing. In that year or later [16] was published, some new approaches appeared [9,10,17]. It is meaningful to summarize or compare those new algorithms. However, this paper aims at establishing the theoretical foundation for the algorithm presented in [16]. We do not present any new algorithm or compare the existing algorithms in this paper.

The rest of the paper is organized as follows: Section 2 discusses the possibilities undershoots occur, the structure of the undershoot set, gives the improved algorithm and studies how the number of undershoot intervals changes. In Section 3, it is proved rigorously that the iterative algorithm converges to an envelope without undershoots with exponential rate of convergence. In Section 4, experiments are conducted to support the result of the convergence rate. Finally, Section 5 gives the conclusion of this paper.

2. Undershoots and elimination

2.1. Undershoots and undershoot intervals

As mentioned in the last section, undershoots often occur when the cubic spline interpolating the maximal/minimal points is employed as the upper/lower envelope of a signal. In this subsection, we will discuss why and how the undershoots occur and then give the structure of the undershoot set. Firstly, we give the definition of undershoots.

Definition 1. Given a signal $x(t)$ defined on $[a, b]$, let $u(t)$ be its upper envelope obtained by using some method. Then the undershoot set is defined by

$$\mathcal{U}(x, u) := \big\{t \in [a, b] : x(t) > u(t)\big\}.$$

If $\mathcal{U}(x, u)$ is an empty set, then the envelope is said to contain no undershoots.

The following theorem reveals the reason and possibility that an envelope contains undershoots.

Theorem 1. *Let $x(t)$ be a differentiable signal defined on $[a, b]$ and $\tau_1 < \cdots < \tau_n$ be all its local maximal points. Let $\{x_j = x(\tau_j) : j = 1, \cdots, n\}$ be its data set and $u(t)$ be a differentiable interpolation function passing $\{(\tau_j, x_j) : j = 1, \cdots, n\}$. For $1 \le i \le n$, denote $d_i = u'(\tau_i)$, then*

(i) *If $d_i \neq 0$, undershoot must occur near τ_i. More exactly, if $d_i > 0$ there must exist $\delta_i > 0$ such that $u(t) < x(t)$ on $(\tau_i - \delta_i, \tau_i)$; else, if $d_i < 0$, there must exist $\delta_i > 0$ such that $u(t) < x(t)$ on $(\tau_i, \tau_i + \delta_i)$.*
(ii) *If $d_i = 0$, undershoots depend on the high-order derivatives of $u(t)$ and $x(t)$ at τ_i if they exist.*

Proof. Using the differentiability of $u(t)$ and $x(t)$ at τ_i and noticing $x'(\tau_i) = 0$ we have that

$$u(t) = u(\tau_i) + d_i(t - \tau_i) + o(t - \tau_i), \qquad x(t) = x(\tau_i) + o(t - \tau_i),$$

where the little-o notation means $o(t - \tau_i)/(t - \tau_i) \to 0$ as $t - \tau_i \to 0$. Therefore

$$u(t) - x(t) = d_i(t - \tau_i) + o(t - \tau_i).$$

The conclusions of the theorem can be easily concluded by the above equality. □

Fig. 2(a)–(e) display graphically some typical cases of undershoots. They do not include all the cases of undershoots. More complicated undershoots may exist, such as the case shown in Fig. 2(f). However, it can be shown that, under a mild condition, the undershoots occur around the local maximal point of the signal if they exist. Detailed discussion will be given in Section 2.3.

The following theorem shows that the undershoot set $\mathcal{U}(x, u)$ is a union of at most countable disjoint open intervals provided that both $x(t)$ and $u(t)$ are continuous on $[a, b]$.

Theorem 2. *Let the upper envelope $u(t)$ defined on $[a, b]$ satisfy $u(a) \ge x(a)$, $u(b) \ge x(b)$. Denote $x_1(t) = x(t) - u(t)$. If both $x(t)$ and $u(t)$ are continuous on $[a, b]$, then the undershoot set can be uniquely expressed as the union of at most countable disjoint intervals $\{(a_i, b_i)\}$:*