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Uncertainty law in ambient modal identification—Part I: Theory

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ABSTRACT

Ambient vibration test has gained increasing popularity in practice as it provides an economical means for modal identification without artificial loading. Since the signal-to-noise ratio cannot be directly controlled, the uncertainty associated with the identified modal parameters is a primary concern. From a scientific point of view, it is of interest to know on what factors the uncertainty depends and what the relationship is. For planning or specification purposes, it is desirable to have an assessment of the test configuration required to achieve a specified accuracy in the modal parameters. For example, what is the minimum data duration to achieve a 30% coefficient of variation (c.o.v.) in the damping ratio? To address these questions, this work investigates the leading order behavior of the 'posterior uncertainties' (i.e., given data) of the modal parameters in a Bayesian identification framework. In the context of well-separated modes, small damping and sufficient data, it is shown rigorously that, among other results, the posterior c.o.v. of the natural frequency and damping ratio are asymptotically equal to $(\zeta/2\pi N_c B_f)^{1/2}$ and $1/(2\pi\zeta N_c B_\zeta)^{1/2}$, respectively; where ζ is the damping ratio; N_c is the data length as a multiple of the natural period; B_f and B_ζ are data length factors that depend only on the bandwidth utilized for identification, for which explicit expressions have been derived. As the Bayesian approach allows full use of information contained in the data, the results are fundamental characteristics of the ambient modal identification problem. This paper develops the main theory. The companion paper investigates the implication of the results and verification with field test data.

1. Introduction

The modal properties of a structure primarily include the natural frequencies, damping ratios and mode shapes. Identifying them based on measured vibration data is an important task often performed in vibration control or structural health monitoring [1–4]. Ambient vibration (output-only) tests have gained increasing popularity in both theory development and practical applications [5–8]. This is to a large extent attributed to its economy in implementation. Ambient vibration data are obtained when the structure is under unknown working load assumed to be random with broadband spectral characteristics. The latter assumptions are required to establish a theoretical stochastic description of the response statistics without knowing the loading time history.

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In ambient vibration tests the loading comes from the stochastic environment whose intensity and frequency content cannot be directly controlled. In the absence of specific loading information, the uncertainty of the identified modal parameters is often significantly larger than that using forced vibration (known input) or free vibration tests where the signal-to-noise (s/n) ratio can be managed to an adequate level. The identified damping ratio, for example, exhibits large variability from one data set to another. The observed variability may come from physical variability, e.g., as a result of thermal or amplitude dependence, or merely statistical variability due to lack of data or modeling error [9–13]. Significant variability can also exist in the mode shape at where the value is small or when the modes are closely-spaced.

It will be useful to assess beforehand the uncertainty in the identified modal properties for given test configuration, although this task can be challenging, recognizing the sophistication of ambient modal identification (operational modal analysis) theories. A Bayesian identification approach allows the 'posterior uncertainty' (i.e., given data) to be calculated for given data and modeling assumptions [14–17]. However, the expressions depend implicitly on the data and they are too generic to give any insights.

This work investigates the leading order behavior of the posterior uncertainties of modal properties identified using ambient vibration data, with the aim to providing insights for managing their uncertainties. A Bayesian FFT approach is adopted that rigorously processes the information available in the data to yield information about the modal parameters. Working in the frequency domain allows a natural extraction of information in the data relevant to the mode and is well suited to time-invariant linear systems with classical modes. This does not introduce any loss of generality by virtue of the one–one correspondence between the time-domain and FFT data. The original formulation first appeared in [18]. Fast equivalent formulations that allow practical implementations are due to [19,8,20]. A recent review can be found in [21].

To keep the problem tractable we focus on the case of well-separated modes, where one can select a frequency band around the natural frequency of interest so that the contribution of response from other modes can be ignored in the identification model. We analyze in detail the posterior covariance matrix of modal parameters and derive its leading order behavior under asymptotic conditions applicable in typical situations, namely, small damping and long data duration. The outcomes are 'asymptotic uncertainty laws' that govern the achievable limits in the accuracy of modal parameters given the modeling assumptions. This paper develops the theory. The companion paper discusses their qualitative aspects, implications and verification with field data.

2. Bayesian modal identification theory

Let the acceleration time history measured at n degrees of freedom (dofs) of a structure be $\{\hat{\ddot{\mathbf{x}}}_j \in R^n : j=1,\dots,N\}$ and abbreviated as $\{\hat{\ddot{\mathbf{x}}}_j\}$, where N is the number of samples per channel. In the context of Bayesian inference, it is modeled as $\hat{\ddot{\mathbf{x}}}_j = \ddot{\mathbf{x}}_j(\boldsymbol{\theta}) + \boldsymbol{\varepsilon}_j$, where $\ddot{\mathbf{x}}_j(\boldsymbol{\theta})$ is the model (theoretical) response that depends on the set of parameters $\boldsymbol{\theta}$ to be identified; $\boldsymbol{\varepsilon}_j$ is the prediction error that accounts for the difference between the model response and data, due to measurement noise and modeling error. The FFT $\{\mathcal{F}_k\}$ of $\{\hat{\ddot{\mathbf{x}}}_j\}$ is defined as

$$\mathcal{F}_k = \sqrt{\frac{2\Delta t}{N}} \sum_{j=1}^{N} \hat{\ddot{\mathbf{x}}}_j \exp\left[-2\pi \mathbf{i}\frac{(k-1)(j-1)}{N}\right] \tag{1}$$

where $\mathbf{i}^2=-1$ and Δt is the sampling interval. For $k=2,\dots,N_q$, $\mathcal{F}_k$ corresponds to frequency $f_k=(k-1)/N\Delta t$, where $N_q=\text{int}[N/2]+1$ (int[.] denotes the integer part) is the index at the Nyquist frequency. In practice, only the $\mathcal{F}_k$'s on a selected frequency band containing the mode(s) of interest is used for identification, which significantly simplifies the identification model. The power spectral density (PSD) of the loading and prediction error need only be flat within the band. This relaxes the conventional white noise assumption, making the method more robust than time-domain methods. Other bands with irrelevant information or which are difficult to model are legitimately ignored, therefore avoiding modeling error. This does not require any signal pre-processing such as filtering or averaging.

Let the structure be classically-damped, i.e., its response can be written as a sum of modal responses that satisfy their own (uncoupled) equation of motion. Assuming a single contributing mode in the selected band, the FFT of the acceleration response in the band is given by $\mathcal{F}_{\ddot{\mathbf{x}}}(k) = \boldsymbol{\Phi}\mathcal{F}_{\ddot{\eta}}(k)$, where $\boldsymbol{\Phi} \in R^n$ is the mode shape confined to the measured dofs assumed to have unit norm, i.e., $||\boldsymbol{\Phi}||^2 = \boldsymbol{\Phi}^T\boldsymbol{\Phi} = 1$; $\mathcal{F}_{\ddot{\eta}}(k)$ is the FFT of the modal acceleration response $\ddot{\eta}$, which satisfies the modal equation of motion

$$\ddot{\eta}(t) + 2\zeta\omega\dot{\eta}(t) + \omega^2\eta(t) = p(t) \tag{2}$$

Here $\omega = 2\pi f$; f is the natural frequency (in Hz), ζ is the damping ratio and $p(t)$ is the modal force (including the modal mass in its denominator) with PSD S within the selected band (i.e., need not be white). The set of parameters $\boldsymbol{\theta}$ to be identified is then given by

$$\boldsymbol{\theta} = \{f, \zeta, S, S_e, \boldsymbol{\Phi}\} \tag{3}$$

where S_e is the PSD of the prediction error within the selected band (i.e., need not be white).

Let $\mathbf{Z}_k = [\mathbf{F}_k; \mathbf{G}_k] \in R^{2n}$, where $\mathbf{F}_k = \text{Re}\mathcal{F}_k$ and $\mathbf{G}_k = \text{Im}\mathcal{F}_k$. The FFT data within the selected band is denoted by $\{\mathbf{Z}_k\}$. Using Bayes' theorem, the posterior probability density function (PDF) of the set of modal parameters $\boldsymbol{\theta}$ given the FFT data $\{\mathbf{Z}_k\}$ is