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High-performance variable band-pass/band-stop state-space digital filters using Gramian-preserving frequency transformation

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ABSTRACT

This paper presents a simple state-space-based method for design and realization of variable band-pass/band-stop IIR digital filters. Our proposed variable filters not only allow us to tune the frequency characteristics, but also ensure high-performance with respect to finite wordlength effects such as L_2-norm dynamic range scaling, limit cycles, roundoff noise, and coefficient sensitivity. We achieve this property using the Gramian-preserving frequency transformation, which is implemented by replacing each delay element in a given prototype filter with a second-order all-pass function that has the four-multiplier-lattice structure. It is shown that our proposed variable filters are described in a rather simple form without the need of the inverse matrix that appeared in the conventional Gramian-preserving frequency transformation. Moreover, we show the high-performance of our proposed method in comparison with other possible types of frequency transformations that are implemented by the typical one-multiplier/two-multiplier-lattice forms and the direct form.

1. Introduction

In many practical applications of digital signal processing, there is a great need for real-time tuning of the characteristics of frequency selective digital filters such as low-pass, high-pass, band-pass, and band-stop filters. Such tunable digital filters are called variable digital filters, and many methods for design of variable digital filters have been proposed [1]. In this paper, we focus on infinite impulse response (IIR) variable digital filters, and the main interest is in design and implementation of variable band-pass filters (VBPFs) and variable band-stop filters (VBSFs).

Design of variable IIR digital filters is classified into two major methods. One is based on the frequency transformation [2–7], and the other is based on the spectral parameters [8–11]. The method based on the frequency transformation is very simple, but this method is known to have a limitation that the tuning can be performed for only the cutoff frequencies of filters. On the other hand, it is argued in [8–11] that the spectral-parameter-based method is more general than the frequency-transformation-based method because the spectral-parameter-based method can deal with much more flexible characteristics. Nevertheless, in this paper we shall focus on the frequency-transformation-based method for the following reasons:

- The frequency-transformation-based variable filters, especially VBPFs and VBSFs, play important roles in the adaptive filtering as well as the filter design. For example, the all-pass-based second-order adaptive notch filters, which are widely used for detection of unknown narrowband signals, are shown to be obtained by the frequency transformation [12]. Another type of frequency-transformation-based adaptive notch filters is also proposed in [13]. Furthermore, very recently we have presented a new framework for adaptive band-pass/band-stop filtering with the help of the frequency transformation [14,15], where we have achieved significant improvement of output signal-to-noise ratio compared with the conventional adaptive notch filtering.
- The frequency transformation has a significant link with the finite wordlength effects of digital filters. This result was earlier proved in [16]: By means of an elegant analysis using the state-space representation, it was proved that the optimal value of the roundoff noise of finite wordlength digital filters is independent of the change in cutoff frequencies. Using this property we have proposed variable low-pass state-space filters of high-performance with respect to the finite wordlength effects [17,18], and thus such variable filters will be very attractive to actual hardware implementations.

In view of these considerations, the main objective of this paper is to present a new method for design and implementation of VBPFs/VBSFs in state-space form, by means of the frequency transformation. Although the state-space approach is a well-known powerful tool for synthesis of optimal filter structures with respect

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to the finite wordlength effects, most of the existing synthesis methods are applicable to only the time-invariant filters. In other words, little has been reported concerning the state-space-based variable digital filters except for our proposed low-pass filters [17,18]. Therefore, the state-space formulation of VBPFs/VBSFs is one of the contributions of this paper. We derive this formulation by applying the Gramian-preserving frequency transformation [19–21] to design and realization of VBPFs/VBSFs. The Gramian-preserving frequency transformation is an extension of the state-space-based frequency transformation given by [16] and it is also known to be important in the field of analog filter design [22]. However, the conventional Gramian-preserving frequency transformation has never investigated its application to design or realization of variable digital filters. Hence application of the theory [19–21] to the practical VBPFs/VBSFs is a new contribution.

Another contribution is the simple mathematical formulation for implementation of VBPFs/VBSFs. It is shown in [20,21] that the conventional Gramian-preserving frequency transformation can be implemented by replacing delay elements in the prototype filter with second-order all-pass filters having the four-multiplier-lattice structure [23]. Although the same strategy is used in this paper, our proposed VBPFs/VBSFs are described in much simpler form than the conventional Gramian-preserving frequency transformation. Our proposed VBPFs/VBSFs are described without the inverse matrix that appeared in the conventional Gramian-preserving frequency transformation.

Furthermore, we present a detailed analysis of the performance of our proposed VBPFs/VBSFs with respect to the finite wordlength effects. In this paper, we discuss the finite wordlength effects from the following well-known criteria: (a) L_2-norm dynamic range scaling, (b) limit cycles, (c) roundoff noise, and (d) coefficient sensitivity. While the theory of [16] discussed only the L_2-norm scaling and the roundoff noise, our analysis takes all of these criteria into account. As a result, we prove that our proposed VBPFs/VBSFs can theoretically attain high-performance with respect to all of these criteria, regardless of the change in the characteristics of VBPFs/VBSFs.

In the above four criteria we pay special attention to (b) limit cycles and (d) coefficient sensitivity, and we present some additional results. Firstly, we derive a new theorem that our proposed VBPFs/VBSFs preserve a condition for the absence of limit cycles. This new theorem includes our earlier work [19–21] about the limit cycles as a special case, and thus our new theorem can be applied to wider classes of filters than our earlier work. In relation to this theorem, we also derive an interesting property that our proposed VBPFs/VBSFs have the unity norm in the coefficient matrix. Secondly, in evaluating the coefficient sensitivity we use two cost functions. One is the conventional Gramian-based sensitivity that is also addressed in our earlier work [19–21]. The other is the L_2-sensitivity that is famous but has never been discussed in our earlier work [19–21]. The evaluation result shows that our proposed VBPFs/VBSFs have high performance with respect to both of the two cost functions.

As another contribution of this paper, for comparison purpose we investigate other alternative types of frequency transformations. Since our proposed method makes use of all-pass filters with the four-multiplier-lattice structure, we also investigate other typical structures—the one-multiplier-lattice structure, the two-multiplier-lattice structure, and the direct form—for the all-pass filters. Consequently, we theoretically and numerically show that our proposed VBPFs/VBSFs outperform all of these three alternatives.

A preliminary version of this paper has been presented at a conference [24], where we just presented mathematical frameworks of our proposed VBPFs/VBSFs, and we roughly discussed their performance. On the other hand, in the present paper we will provide detailed evaluation of our proposed VBPFs/VBSFs. In addition, we will provide a thorough evaluation of our proposed VBPFs/VBSFs in comparison with other lattice-based structures.

Notation. Throughout this paper, we will use the following notations:

- $\Re^{m\times n}$: the set of $m \times n$ real matrices.
- $\boldsymbol{I}_m$: the $m \times m$ identity matrix.
- $\boldsymbol{0}_{m\times n}$: the $m \times n$ zero matrix.
- $\boldsymbol{X}^T$: the transpose of a matrix $\boldsymbol{X}$.
- $(\boldsymbol{X})_{ij}$: the (i, j)-th entry of a matrix $\boldsymbol{X}$.
- $\mathrm{tr}(\boldsymbol{X})$: the trace of a matrix $\boldsymbol{X}$.
- $\|\boldsymbol{X}\|_2$: the 2-induced norm of a matrix $\boldsymbol{X}$.

2. Preliminaries

2.1. State-space representation and its application to high-performance digital filters

Consider the state-space representation of an N-th order digital filter as

$$\boldsymbol{x}(n+1) = \boldsymbol{A}\boldsymbol{x}(n) + \boldsymbol{b}u(n) \tag{1}$$

$$y(n) = \boldsymbol{c}\boldsymbol{x}(n) + du(n) \tag{2}$$

where $u(n)$ and $y(n)$ are the scalar input and the scalar output, $\boldsymbol{x}(n) \in \Re^{N\times 1}$ is the state vector, and $\boldsymbol{A} \in \Re^{N\times N}$, $\boldsymbol{b} \in \Re^{N\times 1}$, $\boldsymbol{c} \in \Re^{1\times N}$ and $d \in \Re^{1\times 1}$ are real-valued coefficients. The transfer function $H(z)$ of this state-space system is given in terms of $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$ by

$$H(z) = d + \boldsymbol{c}(z\boldsymbol{I}_N - \boldsymbol{A})^{-1}\boldsymbol{b}. \tag{3}$$

Throughout this paper, we assume that the state-space filter is asymptotically stable (i.e., the matrix $\boldsymbol{A}$ has all eigenvalues inside the unit circle), and that the state-space filter is controllable and observable.

We next introduce two important matrices that are called the controllability Gramian and the observability Gramian. They are respectively denoted by $\boldsymbol{K}$ and $\boldsymbol{W}$, and they are given as the solutions to the following Lyapunov equations

$$\boldsymbol{K} = \boldsymbol{A}\boldsymbol{K}\boldsymbol{A}^T + \boldsymbol{b}\boldsymbol{b}^T \tag{4}$$

$$\boldsymbol{W} = \boldsymbol{A}^T\boldsymbol{W}\boldsymbol{A} + \boldsymbol{c}^T\boldsymbol{c}. \tag{5}$$

These matrices are known to be symmetric and positive definite.

Here, it is important to note that the set of coefficients $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$ is non-unique for a given transfer function $H(z)$, and that the Gramians $(\boldsymbol{K}, \boldsymbol{W})$ are also non-unique. This means that many different state-space realizations (i.e. many filter structures) exist for a given transfer function $H(z)$, and thus the filter performance with respect to the finite wordlength effects depend on the choice of state-space realizations. Therefore, by choosing an appropriate realization, we can obtain a state-space digital filter that achieves high performance with respect to the finite wordlength effects. Such a high-performance filter can be obtained by optimizing certain cost functions and/or imposing some constraints (or conditions) on state-space realizations. As addressed below, most of such cost functions, constraints and conditions are described in terms of the controllability Gramian $\boldsymbol{K}$ and the observability Gramian $\boldsymbol{W}$.

In this paper, we consider the following criteria on the finite wordlength effects:

(a) (*L_2-norm dynamic range scaling*) The L_2-norm scaling [25,26] is widely applied to digital filters in order to reduce the