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# A time-varying inertia pendulum: Analytical modelling and experimental identification

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ABSTRACT

In this paper two of the main sources of non-stationary dynamics, namely the time-variability and the presence of nonlinearity, are analysed through the analytical and experimental study of a time-varying inertia pendulum. The pendulum undergoes large swinging amplitudes, so that its equation of motion is definitely nonlinear, and hence becomes a nonlinear time-varying system. The analysis is carried out through two subspace-based techniques for the identification of both the linear time-varying system and the nonlinear system.

The flexural and the nonlinear swinging motions of the pendulum are uncoupled and are considered separately: for each of them an analytical model is built for comparisons and the identification procedures are developed. The results demonstrate that a good agreement between the predicted and the identified frequencies can be achieved, for both the considered motions. In particular, the estimates of the swinging frequency are very accurate for the entire domain of possible configurations, in terms of swinging amplitude and mass position.



## 1. Introduction

During the last years, many efforts have been spent in studying non-stationary signals. Within this topic, two fundamental sources of non-stationarity are the time-variability and the presence of nonlinearity.

Classical examples of time-varying systems are absorber springs with non-constant stiffness, or fuel tanks, characterised by mass variation. Another important class of time-varying systems is the case of moving loads: if a structure is travelled by a load whose mass is not negligible with respect to the structure mass, then the dynamical properties of the system change with time. Typical case is a train crossing a railway bridge.

One of the first works on the identification of time-varying systems was conducted by Liu [1,2], where the concept of pseudo-natural frequencies was introduced, that are obtained by the time-varying state transition matrix. Tasker [3] proposed a recursive algorithm, based on subspace methods, to identify the state matrices and successively to determine the modal parameters. Other important approaches are those based on the Kalman Filter [4], or the parametric methods, as for example the FS-TARMA [5], which is an extension of the classical ARMA techniques.

In Ref. [6], a Short-Time Stochastic Subspace Identification (ST-SSI) approach has been defined, based on the "frozen" technique, where the classical subspace identification [7] is applied to successively windowed parts of the signal.

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The ST-SSI method can be applied to different kinds of non-stationary systems in order to estimate the instantaneous frequencies of the signal. For example, it has been used to estimate the frequency in nonlinear systems, as well as in practical systems showing nonlinear effects, such as pre-stressed concrete beams tested in laboratory, in which a softening nonlinearity was identified [8]. In a similar way, an input–output linear system subspace identification method has been applied in Ref. [9] for characterisation of nonlinear dynamical structural systems: time-varying instantaneous natural frequencies and mode shapes of nonlinear civil structures are extracted.

However, instead of extracting a series of time-varying linear models, the identification of a whole parametric nonlinear model is an important instrument for many purposes. For example, it would allow for treating nonlinearities in possibly damaged structures [10], or attaining improved predictions of vibration response amplitude, which is an issue for accurate long term fatigue estimates. Any nonlinear method is requested to correctly identify the classes of nonlinearity present and possibly to quantify the extent of their force contributions. Nonlinear system identification has been thoroughly investigated in recent years and many efforts have been spent leading to a large number of methods. A comprehensive list describing the past and recent developments is given in Ref. [11].

Among them, the Conditioned Reverse Path (CRP) method [12,13] is based on the construction of a hierarchy of uncorrelated response components in the frequency domain, allowing the estimation of the coefficients of the nonlinearities away from the location of the applied excitation.

The Nonlinear Identification through Feedback of the Outputs (NIFO) [14] is a frequency domain method which has demonstrated some advantages with respect to the CRP, mainly due to the lighter conceptual and computing effort. This method exploits the spatial information and interprets nonlinear forces as unmeasured internal feedback forces.

Starting from the basic idea of NIFO, the Nonlinear Subspace Identification (NSI) method has been developed [15] and improved [16] for identifying large systems with lumped nonlinearities. NSI is a time domain method which exploits the robustness and the high numerical performances of the subspace algorithms. In order to extend the NSI method to be applied also on realistic large nonlinear engineering structures, a modal counterpart has been developed in Ref. [17].

In the present paper, both the ST-SSI and the NSI methods are applied to the experimental case of a pendulum with time-varying inertia. Moreover, a nonlinear behaviour is expected, since the swinging angles are large.

The analysis and simulation [18] of mechanical systems with imposed relative motion of components is challenging: time-varying inertia, created by a part that slides along a rotating member, reveals the Coriolis-type effects present in the system. This relative movement can excite but also reduce the structure vibration, providing new means or techniques for active attenuation of vibrations. An example of such a technique, in which a mass moving radially is treated as a controller to attenuate the pendulum swings, was demonstrated in Ref. [19].

The concept of controlling the motion of a system through mass reconfiguration has been examined in Ref. [20] using a variable length mathematical pendulum. The control of angular oscillations is accomplished by sliding the end mass towards and away from the pivot. A variable length pendulum has also been considered in Ref. [21], where a rigorous qualitative investigation of its equation is carried out without any assumption on small swinging amplitudes. The exact and approximate study of the nonlinear pendulum can be found in various recent papers; most of them deal with obtaining analytical approximate expressions for the large-angle pendulum period [22,23]. Among the few papers devoted to obtaining approximate solutions (the angular displacement as a function of time), Ref. [24] derives an accurate expression in terms of elementary functions.

The paper starts with an overview of the ST-SSI and the NSI methods; then, in Section 3, the experimental setup is described and the fundamental relationships about the dynamics of the pendulum are extracted. In Section 4, a model for the pendulum is proposed for the flexural vibrations and successively the identified frequencies are compared to those obtained by the Rayleigh–Ritz approach. In Section 5, the swing motion of the pendulum is considered, firstly with a fixed mass and then with a moving mass travelling on it. The final results show that the identified swinging frequency and the theoretical one are very similar.

## 2. Methodology

Before discussing the experimental application, it is necessary to briefly introduce the identification methods that can be used in this particular case. To the authors' knowledge, there are no methodologies able to perform a reliable identification by taking the two effects into account, therefore an explanation of two methods, one specific for linear time-varying systems and one ad hoc for nonlinear systems, is proposed.

Both the presented procedures are based on the subspace methods introduced by Van Overschee and De Moor [7].

### 2.1. ST-SSI method

The procedure for the identification of linear time-varying systems is called Short-Time Stochastic Subspace Identification (ST-SSI) [6,25]. The idea is to divide the signal in many parts and consider the system as time-invariant in each time interval: the process is called the frozen technique.

If the output data are measured at discrete times with a sampling interval $\Delta t$ and the input is a discrete signal characterised by a zero-order hold between consecutive sample points, the corresponding discrete-time state-space