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Nonorthogonal bases and phase decomposition: Properties and applications

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ABSTRACT

In a previous paper [1] it was discussed the viability of functional analysis using as a basis a couple of generic functions, and hence vectorial decomposition. Here we complete the paradigm exploiting one of the analysis methodologies developed there, but applied to phase coordinates, so needing only one function as a basis. It will be shown that, thanks to the novel iterative analysis, any function satisfying a rather loose requisite is ontologically a basis. This in turn generalizes the polar version of the Fourier theorem to an ample class of nonorthogonal bases. The main advantage of this generalization is that it inherits some of the properties of the original Fourier theorem. As a result the new transform has a wide range of applications and some remarkable consequences. The new tool will be compared with wavelets and frames. Examples of analysis and reconstruction of functions using the developed algorithms and generic bases will be given. Some of the properties, and applications that can promptly benefit from the theory, will be discussed. The implementation of a matched filter for noise suppression will be used as an example of the potential of the theory.

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1. Introduction

The Fourier theorem is one of the most valuable mathematical tool, it is used in all sort of applications, from mobile phones to system theory. In its standard form the theorem states the possibility to reconstruct a function using a series of sines and cosines. Over the years different generalizations have been devised for the theorem. One of the most useful is the use of bases other than the orthogonal pair, for example using wavelets and frames. Alas these tools for nonorthogonal bases sport a high computational complexity, tolerate only few specially built functions as bases, and require different bases for analysis and reconstruction (biorthogonal and dual bases) [6–9]. These constraints have limited the diffusion of the wavelets and frames to special applications, compared to the pervasiveness of the original Fourier theorem.

A recent development is the common waveform analysis [10,14,15]. There, a couple of even and odd special functions as the square wave, triangular and the like constitutes the basis. However, due to the limitation of the mathematics involved (that is based on an inner product computation), this theory still requires the use of biorthogonal bases, works only with a limited number of special bases and vectorial decomposition.

A different approach has been introduced in [1]. There, two procedures were developed for the analysis. One method has been called "direct" or brute force and requires the solution of a system of equations (much like the algorithm used in frame theory). The second method has been dubbed "indirect" because uses a novel iterative algorithm. The two methods have been used together in [1] to validate each other, although it has been briefly illustrated the superiority of the indirect method over the direct one.

The theory developed in [1], compared to the known tools as the wavelets, the frames and the common waveform analysis, admits a much larger class of functions as bases and, most notably, utilizes the same basis for analysis and reconstruction. All thanks just to a change in the analysis paradigm. The idea is that there is no reason why we should limit ourselves to the use of the inner product in the analysis. In effect the definition of basis does not mention the analysis phase, requiring only the possibility of reconstruction of any function of the given space in terms of a combination of the basis (plus the usual constraints of unicity and convergence of the reconstruction). In other words we are left free to choose the analysis method (here we prefer the word "decomposition" for reasons that will be clear below). Traditionally this freedom is not exploited as the vast majority of the established algorithms for nonorthogonal bases employ the same inner product computation of the original Fourier theorem. And thus a "direct" approach.

The advantage is the possibility of calculating any of the components independently, but at the same time, oblige to use biorthogonal functions when the basis is nonorthogonal and greatly limits the choice of the basis. Instead the recursive computation methodology (the "indirect" method) exploits an iterative

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change of coordinates between the orthogonal basis and the new, nonorthogonal, one [1].

To find the components of a signal with the indirect methodology, one has to start with the decomposition of the signal in the usual orthogonal basis. This generates a representation in the Hilbert space. Then an iterative algorithm for change of reference translates these components from the Hilbert space to the new basis. The originality of the method is that we do not try to find directly the components of a function on a nonorthogonal basis. Instead, we switch from a known representation (the Fourier components in the Hilbert space) to a new, equivalent, one. For the transitive property of equality, if one is a representation of the signal so is the other.

In the literature there is another main iterative algorithm for function analysis: the Empirical Mode Decomposition (EMD), as used for example in Hilbert–Huang transform [17]. However there are substantial differences between this approach and the EMD. For one EMD is, by definition, empirical: the basis functions are derived from the data. Instead here, one first chooses the basis, then the parameters of the decomposition are computed. The benefit is that one can pick the basis that is best suited to a particular problem.

Matching Pursuit (MP) [12] is another common algorithm for function analysis. There a dictionary, generally consisting of a large collection of time–frequency atoms, is employed in the search for the best sparse representation of a signal, minimizing the error. The tool is most useful for compression and coding but is very computing intensive. Instead in [1] there is much more freedom in the choice of the basis, and the decomposition is purely in frequency. And for each basis an exact representation of the signal is computed up to any chosen frequency. In other words the error can be confined at highest frequencies and with a very efficient algorithm.

An added advantage of the methodology is that one can still use the metric of the Hilbert space to assert the convergence of the procedure. The only drawback of the iterative method is that to find the component at a given frequency, generally all the components at lower frequency must be computed. But we think that this is not an issue given the possibilities that the new theory opens up.

A further advantage is that one can now use the same basis for analysis and reconstruction (in contrast with the usual tools for nonorthogonal bases).

But probably the main benefit of the “indirect methodology” is that it is applicable even to polar decomposition and this will help us to surmount the borders of linear dependence.

As a matter of fact all of the cited tools are based on the methods of linear algebra, so inherently vectorial. However, at the time of interpreting the analysis results, these tools are no match for the simplicity of a Fourier power spectrum in terms of amplitude and phase.

The natural extrapolation of the theory developed in [1] is thus applying the same computing method to phase decomposition. The benefit of phase (in orthogonal terms: “polar”) decomposition is in dealing with a single function as a basis. And a single function as a basis has the indisputable advantage (in contrast with the vectorial tools) that some of the properties of the Fourier theorem can be extended also to nonorthogonal bases, and this greatly enhances the applicability of the tool, as it will be clear in the following.

When orthogonal bases are involved, polar and vectorial representations are essentially the same thing, as there is a trivial equation connecting the two coordinate systems. Instead, with nonorthogonal bases, the vectorial and phase decompositions of a function are completely different beasts, and there is no simple way to pass from one to the other. An example could clarify the point. Imagine of having as a function a square wave with arbitrary phase. If we use vectorial decomposition and a basis consisting of even and odd square waves, a viable basis according to [1,10,14,15] (they all give the same results, although the last papers are based on traditional approach and biorthogonal bases), it is evident that we would need an infinite series of square waves to reconstruct the function. Because the nonorthogonal vectorial decomposition cannot easily characterize the arbitrary phase. Instead, when using phase decomposition and a single square wave as a basis [13], the result of the analysis is a single couple of parameters at the given frequency: amplitude and phase. Much more efficient and comprehensible.

The disadvantage of the phase decomposition is that its outcome is a set of two parameters: amplitude and phase (or more generally shift) that are not homogeneous, differing dimensionally, and hence preventing the use of matrices and linear algebra in the computations (that would be precluded anyway because in case of nonorthogonal bases the resulting systems will be nonlinear, as it will be shown below). As a consequence, for orthogonal bases the vectorial analysis is preferred, as in the most common flavors of the Fourier theorem. Whereas, when using nonorthogonal bases, the phase decomposition is more widely applicable and delivers more interesting results, even with the added burden of dealing with couples of non-homogeneous parameters.

A previous paper [13] introduced the phase decomposition over nonorthogonal bases but with a focus on a special application. There, it was demonstrated that the square wave is one of the viable bases for phase decomposition. As the square wave is the natural output of digital systems, it was exploited in the design of very efficient, multiplierless, signal synthesizers. The systems employing the square wave are very frugal on computing demands and suitable for many applications.

The goal of this article is to disclose the bigger picture, revealing some of the properties and consequences of the theory of phase decomposition over nonorthogonal bases and indicating other applications.

2. The iterative analysis methodology

Here the rationale behind the new computation scheme will be briefly summarized in order to introduce a fast analysis algorithm and other consequences of the decomposition.

Lemma. *Given a Hilbert space $\boldsymbol{H}$ with the usual orthonormal basis, then any function $S(x) \in \boldsymbol{H}$ spans the space when using the same frequency–phase reconstruction algorithm of the polar Fourier decomposition. I.e. $\{S(nx)\}$ is complete (n being the frequency).*

We shall prove the feasibility for real periodic functions $f(x)$, $S(x) \in L^2[-\pi, +\pi]$ (the space of periodic Lebesgue square integrable functions). While the extension to complex valued functions, different periods and transforms is straightforward.

Given any periodic function $f(x) \in L^2[-\pi, +\pi]$ satisfying the Dirichlet conditions, it can be expressed as a Fourier series. We omit here an eventual average (a DC component) from the series as it is a simple constant that will not change our conclusions:

$$f(x) = \sum_{k=1}^{\infty} b_k \cos(kx + \vartheta_k) \tag{1}$$

and given another nonzero periodic function with Fourier series:

$$S(x) = \sum_{p=1}^{\infty} s_p \cos(px + \phi_p) \tag{2}$$

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