Available online at www.sciencedirect.com

ScienceDirect

Digital Signal Processing 16 (2006) 498–522

Digital Signal Processing

www.elsevier.com/locate/dsp

A statistical analysis of the Delogne–Kåsa method for fitting circles ☆

Emanuel E. Zelniker *,1, I. Vaughan L. Clarkson

School of Information Technology and Electrical Engineering, The University of Queensland, Queensland 4072, Australia

Available online 26 April 2005

Abstract

In this paper, we examine the problem of fitting a circle to a set of noisy measurements of points on the circle's circumference. Delogne [Proc. IMEKO-Symp. Microwave Measurements, 1972, pp. 117–123] has proposed an estimator which has been shown by Kåsa [IEEE Trans. Instrum. Meas. 25 (1976) 8–14] to be convenient for its ease of analysis and computation. Using Chan's circular functional model to describe the distribution of points, we perform a statistical analysis of the estimate of the circle's centre, assuming independent, identically distributed Gaussian measurement errors. We examine the existence of the mean and variance of the estimator for fixed sample sizes. We find that the mean exists when the number of sample points is greater than 3 and the variance exists when this number is greater than 4. We also derive approximations for the mean and variance for fixed sample sizes when the noise variance is small. We find that the bias approaches zero as the noise variance diminishes and that the variance approaches the Cramér–Rao lower bound.

Keywords: Circle fitting; Least squares; Maximum-likelihood estimation; Moore–Penrose inverse; Cramér–Rao lower bound; Random matrices; Wishart distribution

☆ A preliminary version of this paper was presented at the 2003 IEEE International Symposium on Signal Processing and Information Technology, December 14–17, Darmstadt, Germany.

* Corresponding author.

E-mail address: zelniker@itee.uq.edu.au (E.E. Zelniker).

[1] E.E. Zelniker is additionally supported by a scholarship from the Commonwealth Scientific & Industrial Research Organisation.

1051-2004/$ – see front matter
doi:10.1016/j.dsp.2005.04.001

1. Introduction

The accurate fitting of a circle to noisy measurements of points on its circumference is an important and much-studied problem in statistics. It has applications in many areas of research including archaeology [1], geodesy [2], physics [3,4], microwave engineering [5] and computer vision and metrology [6].

The problem of obtaining an accurate circular fit, by which we mean the estimation of a circle's centre and its radius, appears to have been first studied by Thom [1] in connection with measurements of ancient stone circles in Britain. He proposes an approximate method of least-squares solution. In addressing a problem of 'statistical geography,' Robinson [2] gives a complete formulation of the solution to the problem by the method of least squares.

The first detailed statistical analysis to be published appears to be that of Chan [7]. He proposes a 'circular functional relationship,' which we also use as the basis for our investigations. In this model, it is assumed that the measurement errors are instances of independent and identically distributed (i.i.d.) random variables. Additionally, the points are assumed to lie at fixed but unknown angles around the circumference, i.e., not only are the centre and radius of the circle unknown parameters to be estimated, but so are the angles of each circumferential point. He derives a method to find the maximum-likelihood estimator (MLE) when the errors have a Gaussian distribution. This method is identical to the least-squares method of [2]. He also examines the consistency of the estimator.

A disadvantage of the MLE is that it is difficult to analyse. From a numerical point of view, another disadvantage is that the only known algorithms for computing the MLE are iterative. Furthermore, it is known that there are instances in which there is no minimum, but rather a stationary point, or several local minima in the likelihood function [8,9]. The difficulties with the MLE were recognised by Kåsa [5], who proposes using a simple estimator due to Delogne [10] which is relatively easy to analyse and also to compute. This estimator has subsequently been independently rediscovered at least four times [11–14]. It has also been analysed in situations where the noisy data points are crowded together [15].

Berman and Culpin [8] have carried out a detailed statistical analysis of both the MLE and the Delogne–Kåsa estimator (DKE). Specifically, they prove some results regarding the asymptotic consistency and variance of the estimates. Chan and Thomas [16] have investigated the Cramér–Rao lower bound (CRLB) for estimation in the circular functional model, but see also [17].

In this paper, we are interested in the properties of the DKE for fixed (small) sample sizes rather than its asymptotic properties. Kåsa himself carries out a 'first-order' or 'small-error' analysis of the estimator [5]. However, when the random variables that give rise to the errors are Gaussian, it can no longer be guaranteed that the errors will always be small, no matter how small the variance, and so the analysis becomes invalid. At the outset, it is not even clear whether the mean or variance of the estimator exists.

Kåsa states, by way of justifying the first-order analysis, that 'it may be appreciated that [the expressions for the mean and variance of the estimator] are, in general, very hard to evaluate.' Nevertheless, in this paper, we demonstrate that, under certain conditions, the defining integrals are not wholly intractable. From analysis of the integrals, we set out conditions for which the mean and variance of the DKE for circle centre exist for fixed sample sizes under Chan's circular functional model with Gaussian errors. Where the mean