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Full Length Article

Some applications of $D\alpha$-closed sets in topological spaces

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ABSTRACT

In this paper, a new kind of sets called $D\alpha$-open sets are introduced and studied in a topological spaces. The class of all $D\alpha$-open sets is strictly between the class of all α-open sets and g-open sets. Also, as applications we introduce and study $D\alpha$-continuous, $D\alpha$-open, and $D\alpha$-closed functions between topological spaces. Finally, some properties of $D\alpha$-closed and strongly $D\alpha$-closed graphs are investigated.

1. Introduction and preliminaries

Generalized open sets play a very important role in General Topology, and they are now the research topics of many topologies worldwide. Indeed a significant theme in General Topology and Real Analysis is the study of variously modified forms of continuity, separation axioms, etc. by utilizing generalized open sets. One of the most well-known notions and also inspiration source are the notion of α-open [1] sets introduced by Njåstad in 1965 and generalized closed (g-closed) subset of a topological space [2] introduced by Levine in 1970.

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Since then, many mathematicians turned their attention to the generalization of various concepts in General Topology by considering α-open sets [3–10] and generalized closed sets [11–13]. In 1982 Dunham [14] used the generalized closed sets to define a new closure operator, and thus a new topology τ^*, on the space, and examined some of the properties of this new topology. Throughout the present paper $(X, \tau), (Y, \sigma)$ and (Z, ν) denote topological spaces (briefly X, Y and Z) and no separation axioms are assumed on the spaces unless explicitly stated. For a subset A of a space (X, τ), $Cl(A)$ and $Int(A)$ denote the closure and the interior of A, respectively. Since we require the following known definitions, notations, and some properties, we recall in this section.

Definition 1.1. Let (X, τ) be a topological space and $A \subseteq X$. Then

(i) A is α-open [1] if $A \subseteq Int(Cl(Int(A))$ and α-closed [1] if $Cl(Int(Cl(A)) \subseteq A$.
(ii) A is generalized closed (briefly g-closed) [2] if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.
(iii) A is generalized open(briefly g-open) [2] if $X \backslash A$ is g-closed.

The α-closure of a subset A of X [3] is the intersection of all α-closed sets containing A and is denoted by $Cl_\alpha(A)$. The α-interior of a subset A of X [3] is the union of all α-open sets contained in A and is denoted by $Int_\alpha(A)$. The intersection of all g-closed sets containing A [14] is called the g-closure of A and denoted by $Cl^*(A)$, and the g-interior of A [15] is the union of all g-open sets contained in A and is denoted by $Int^*(A)$.

We need the following notations:

- $\alpha O(X)$ (resp. $\alpha C(X)$) denotes the family of all α-open sets (resp. α-closed sets) in (X, τ).
- $GO(X)$ (resp. $GC(X)$) denotes the family of all generalized open sets (resp. generalized closed sets) in (X, τ).
- $\alpha O(X, x) = \{U \mid x \in U \in \alpha O(X, \tau)\}$, $O(X, x) = \{U \mid x \in U \in \tau\}$ and $\alpha C(X, x) = \{U \mid x \in U \in \alpha C(X, \tau)\}$.

Definition 1.2. A function $f : X \to Y$ is said to be:

(i) α-continuous [16] (resp. g-continuous [17]) if the inverse image of each open set in Y is α-open (resp. g-open) in X.
(ii) α-open [16] (resp. α-closed [16]) if the image of each open (resp. closed) set in X is α-open (resp. α-closed) in Y.
(iii) g-open [18] (resp. g-closed [18]) if the image of each open (resp. closed) set in X is g-open (resp. g-closed) in Y.

Definition 1.3. Let $f : X \to Y$ be a function:

(i) The subset $\{(x, f(x)) \mid x \in X\}$ of the product space $X \times Y$ is called the graph of f [19] and is usually denoted by $G(f)$.
(ii) a closed graph [19] if its graph $G(f)$ is closed sets in the product space $X \times Y$.
(iii) a strongly closed graph [20] if for each point $(x, y) \notin G(f)$, there exist open sets $U \subset X$ and $V \subset Y$ containing x and y, respectively, such that $(U \times Cl(V)) \cap G(f) = \phi$.
(iv) a strongly α-closed graph [21] if for each $(x, y) \in (X \times Y) \backslash G(f)$, there exist $U \in \alpha O(X, x)$ and $V \in O(Y, y)$ such that $(U \times Cl(V)) \cap G(f) = \phi$.

Definition 1.4. A topological space (X, τ) is said to be:

(i) α-T_1 [9] (resp. g-T_1 [22]) if for any distinct pair of points x and y in X, there exist α-open (resp. g-open) set U in X containing x but not y and an α-open (resp. g-open) set V in X containing y but not x.
(ii) α-T_2 [8] (resp. g-T_2 [22]) if for any distinct pair of points x and y in X, there exist α-open (resp. g-open) sets U and V in X containing x and y, respectively, such that $U \cap V = \phi$.

Lemma 1.5. Let $A \subseteq X$, then

(i) $X \setminus Cl^*(A) = Int^*(X \setminus A)$.
(ii) $X \setminus Int^*(A) = Cl^*(X \setminus A)$.

Lemma 1.6. A function $f : (X, \tau) \to (Y, \sigma)$ has a closed graph [19] if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in O(X, x)$ and $V \in O(Y, y)$ such that $f(U) \cap V = \phi$.

Lemma 1.7. The graph $G(f)$ is strongly closed [23] if and only if for each point $(x, y) \notin G(f)$, there exist open sets $U \subset X$ and $V \subset Y$ containing x and y, respectively, such that $f(U) \cap Cl(V) = \phi$.

2. $D\alpha$-closed sets

In this section we introduce $D\alpha$-closed sets and investigate some of their basic properties.

Definition 2.1. A subset A of a space X is called $D\alpha$-closed if $Cl^*(Int(Cl^*(A))) \subseteq A$.

The collection of all $D\alpha$-closed sets in X is denoted by $D\alpha C(X)$.

Lemma 2.2. If there exists an g-closed set F such that $Cl^*(Int(F)) \subseteq A \subseteq F$, then A is $D\alpha$-closed.

Proof. Since F is g-closed, $Cl^*(F) = F$. Therefore, $Cl^*(Int(Cl^*(A))) \subseteq Cl^*(Int(Cl^*(F))) = Cl^*(Int(F)) \subseteq A$. Hence A is $D\alpha$-closed.

Remark 2.3. The converse of above lemma is not true as shown in the following example.

Example 2.4. Let (X, τ) be a topological space, where $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. Then $F_X = \{\phi, \{c\}, \{b, c\}, X\}$, $GC(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$, $GO(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, $D\alpha C(X) = \{\phi, \{b\}, \{c\}, \{a, c\}, \{b, c\}, X\}$, $D\alpha O(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Therefore $\{c\} \in D\alpha C(X)$ and $\{a, c\} \in GC(X)$ but $Cl^*(Int\{a, c\}) = \{a, c\} \not\subset \{c\} \subset \{a, c\}$.

Theorem 2.5. Let (X, τ) be a topological space. Then

(i) Every α-closed subset of (X, τ) is $D\alpha$-closed.
(ii) Every g-closed subset of (X, τ) is $D\alpha$-closed.

Proof. (i) Since closed set is g-closed, $Cl^*(A) \subseteq Cl(A)$ [14]. Now, suppose A is α-closed in X, then $Cl(Int(Cl(A))) \subseteq A$. Therefore, $Cl^*(Int(Cl^*(A))) \subseteq Cl(Int(Cl(A))) \subseteq A$. Hence A is $D\alpha$-closed in X.