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Closed form solution to the optimality equations of minimal norm actuation ☆

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ABSTRACT

The paper deals with natural frequency assignment of vibrating systems by single-input feedback control. A closed form solution to the problem of selecting the input vector which leads to minimal norm of the control gain vector is given. The solution may be applied in applications to reduce the control effort.

1. Introduction

We consider the problem of minimal norm actuation in the context of natural frequency assignment by feedback control. For simplicity, the natural oscillations of the system under consideration are governed by the set of second order differential equations

$$\ddot{\mathbf{x}}+\mathbf{K}\mathbf{x}=\mathbf{0}, \tag{1}$$

where the stiffness matrix $\mathbf{K}\in\Re^{n\times n}$ is a symmetric semi-positive definite matrix and the mass matrix $\mathbf{M}=\mathbf{I}$. If $\mathbf{M}\neq\mathbf{I}$, but it is symmetric positive definite, then by proper transformation the equations of motion may be taken in the form of (1).

The dynamics of the system may be altered by design with feedback control force, so that the motion of the controlled system is governed by,

$$\ddot{\mathbf{z}}+\mathbf{K}\mathbf{z}=\mathbf{b}\mathbf{g}^T\mathbf{z},\quad \mathbf{b},\mathbf{g}=\Re^n. \tag{2}$$

The systems (1) and (2) are called the open loop and the closed loop systems, respectively.

Separation of variables

$$x(t)=\mathbf{v}\ \sin\ \omega t, \tag{3}$$

where $\mathbf{v}$ is a constant vector, applied to (1) gives

$$(\mathbf{K}-\lambda\mathbf{I})\mathbf{v}=\mathbf{0},\quad \lambda=\omega^2. \tag{4}$$

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The symmetric eigenvalue problem (4) has n real positive eigenvalues λ_i with corresponding eigenvectors $\mathbf{v}_i \neq \mathbf{0}$ which solve the problem. The eigenvectors $\mathbf{v}_i$, $i=1,2,\ ...,\ n$, may be chosen to form an orthogonal set. For the sake of definiteness we will scale eigenvectors to have unit norm, hence,

$$\mathbf{v}_i^T\mathbf{v}_j = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}, \quad i,j=1,2,\ ...,\ n. \tag{5}$$

Separation of variables

$$\mathbf{z}(t)=\mathbf{w}\ \sin\ \gamma t, \tag{6}$$

applied to (2) gives

$$(\mathbf{K}-\mu\mathbf{I})\mathbf{w}=\mathbf{b}\mathbf{g}^T\mathbf{w}, \quad \mu=\gamma^2. \tag{7}$$

In general, the non-symmetric eigenvalue problem (7) has, over the complex field, n eigenvalues μ_i with corresponding eigenvectors $\mathbf{w}_i$ which solve (7).

With the control force $\mathbf{b}\mathbf{g}^T\mathbf{z}(t)$ each natural frequency ω_i of the open loop system (1) may be assigned to a desired positive natural frequency $\gamma_i \in \Re$ of the closed-loop system (2), provided that $\mathbf{b}^T\mathbf{v}_i \neq 0$. If $\mathbf{b}^T\mathbf{v}_i=0$ then the natural frequency ω_i is invariant under the control, i.e., $\gamma_i=\omega_i$.

Guzzardo et al. have considered in [6] the problem of determining the input vector $\mathbf{b}$ and the control gain vectors $\mathbf{g}$ which minimizes the second norm $|\mathbf{b}\mathbf{g}^T|$. This problem was referred in the opening sentence of the paper as the *minimal norm actuation in context of natural frequency assignment*. A closed form solution is given in [6] to the problem when a single natural frequency is changed desirably while keeping the rest of natural frequencies unchanged. The equations of optimality for the general case where several natural frequencies are altered are also developed in [6] but they were not solved there. The objective of our paper is to present a closed form solution to the problem in the setting of (1).

The problem of minimal norm actuation by using multi input control was studied by Bai et al. [1,2], and Carvalho and Datta [3]. The approach in these papers was to exploit the redundancy in parameters offered by the multi input control to obtain minimal norm actuation for a given input matrix. We employ the single input case and solve the non linear problem of finding the input vector that achieves the optimization goal. More complete treatment of the subject is presented in the Ph.D. dissertation of Guzzardo [7] and the Master thesis of Guerra [5].

The paper is organized as follows. The partial natural frequencies assignment problem is defined in Section 2. The closed form solution to the optimal actuation problem is given in Section 3. Section 4 includes numerical examples and the paper is concluded in Section 5.

2. Partial natural frequencies assignment

Datta et al. [4] gave an explicit solution to problem of partial pole assignment in the symmetric quadratic eigenvalue problem

$$(s^2\mathbf{M}+s\mathbf{C}+\mathbf{K})\mathbf{w}=\mathbf{b}\left(s\mathbf{f}^T+\mathbf{g}^T\right)\mathbf{w}, \tag{8}$$

where a predesigned self-conjugate subset of poles s_i is assigned by the control $\mathbf{b}\left(s\mathbf{f}^T+\mathbf{g}^T\right)\mathbf{w}$ to a predetermined self-conjugate set, while keeping the rest of the spectrum unchanged. Their solution is reduced to the problem studied here in the following Lemma.

Lemma 1. *With*

$$\mathbf{g}=\sum_{k=1}^{m}\vartheta_k\mathbf{v}_k \tag{9}$$

where

$$\vartheta_k=\frac{\lambda_k-\mu_k}{\mathbf{b}^T\mathbf{v}_k}\prod_{\substack{i=1\\ i\neq k}}^{m}\frac{\lambda_k-\mu_i}{\lambda_k-\lambda_i}, \tag{10}$$

the eigenvalues of (7) *are*

$$\{\mu_k\}=\left\{\mu_1\ \cdots\ \mu_m\ \lambda_{m+1}\ \cdots\ \lambda_n\right\}. \quad \square \tag{11}$$

Define

$$\xi_k \equiv (\lambda_k-\mu_k)\prod_{\substack{i=1\\ i\neq k}}^{m}\frac{\lambda_k-\mu_i}{\lambda_k-\lambda_i}, \tag{12}$$