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Approximate inversion formula for structural dynamics and control

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ABSTRACT

In a recent paper the authors presented an iterative method and an approximate formula for predicting the response of a system modified by a finite set of rank-one modifications. This method was developed based on the successive application of the Sherman–Morrison matrix inversion formula to calculate the inverse of a matrix changed by a rank-k modification and provides an easier method to calculate the change in the transfer matrix of a dynamic system when it undergoes a number of k simply connected modifications. This paper presents a new and more interesting justification of the method which allows an extension of the approximate formula from a formula for frequency response approximation to a more general approximate matrix inversion formula applicable to particularly shaped matrices generally encountered in structural dynamics and control. This new approach extends the area of the application of the previously presented method. An example of a simple application of this method to the problem of feedback control of structures known only through their estimated receptances is presented.

1. Introduction

The study of the structural modification of dynamical systems is of major interest in areas such as system dynamics and control. The direct structural modification problem consists of predicting the dynamic behaviour of a structure after applying a modification using the response of the unmodified structure. The advantages of this method are two-fold: first it requires a smaller computational effort and secondly in the case of using the measured structural response of the initial structure it does not require an analytical model. Many of the studies concerning the structural modifications have as objective the poles or zeros assignment by introducing a passive modification to a structure [1,2]. In [3] the possibility of predicting the behaviour of a structure as a result of changes in parameter values was studied. The most important aspect of the problem, in what concerns this study is that from a mathematical point of view the modification problem consists of a unit or low-rank modification of a known matrix which describes the dynamics of the unmodified structure.

The static counterpart of the structural modification problem is usually called structural reanalysis. In most of the practical cases the stiffness matrix of a structure is changed by a small modification which in turn generates a small modification of the response. A couple of methods used for reanalysis are based on an explicit relationship between the modified inverse, the original inverse and the changes in the stiffness matrix [4]. Akgun [5] studied some of the reanalysis methods and concluded that many of them are in fact based on the low-rank matrix modification formulae which will be discussed below.

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The problem of structural modification introduced by attaching to a primary system a set of simply connected structures was studied in [6] as theoretical basis for a design methodology applied to multiple tuned mass dampers. Like many of the exact methods used for structural modification, the method used in [6] focused on the low-rank modifications where exact solutions can be found at a reasonable computational cost. The solution presented was based on the matrix inversion formulae [7–9] which presented a way of determining directly the inverse of a matrix modified by a low-rank matrix by updating the inverse of the initial matrix. This approach places the problem in a more general mathematical framework which extends well beyond the area of structural analysis. The importance of these formulae is recognised in areas of optimisation, parameter estimation and statistics.

The method of updating the inverse of a matrix modified by the changes of elements in one row or column was discussed by Sherman and Morrison [7] and then by Bartlett [8] who proposed a more general matrix formula for the rank-one modification and applied it to inversion problems arising in discriminant analysis.

The Sherman–Morrison–Woodbury matrix identity formula [9] extends the application of the method and provides a solution for the inverse of a matrix after a low-rank perturbation in terms of the inverse of the original matrix and the perturbation. The most important advantage brought about by the use of the matrix identity formula is that the rank of the modification which is usually much smaller than the dimension of the modified matrix is the dimension of the matrix which needs to be inverted. This leads to an important reduction of the computational cost but still a matrix inversion has to be handled.

The connection between the Sherman–Morrison and Sherman–Morrison–Woodbury formulae was investigated in several studies [5,6] where it was pointed out that in cases particular to structural analysis with an iterative application of Sherman–Morrison formula it is actually possible to update the inverse of a matrix modified by a rank k-modification. The advantage of using the Sherman–Morrison inversion formula is that the only inverse used is the inverse of the original matrix which in structural dynamics can be the receptance matrix of a structure which can be directly estimated from experiment.

It was shown in [6] that this iterative use of a rank one modification can be productive in designing structures with multiple tuned mass dampers or by extension with multiple independent feedback controllers. The iterative use of Sherman–Morrison formula could be substantially simplified and the existence of an approximation formula was put forward. This approximation method related directly every modification to the initial system transfer function which can be experimentally estimated. As examples, two Euler–Bernoulli beams in two set-ups were studied. The modifications consisted of a set of tuned mass dampers attached at the same location or at different locations on the beam.

It should be highlighted that the theory presented in [6] is representative of the case of simply connected structures and a system modified by a set of tuned mass dampers is just a particular case. In this category not only the tuned mass damper but also any simply connected multiple degree of freedom system [10] could constitute an example of modification. In a more general framework it can be stated that any structure acted on by a set of point modifications represented as forces could be suited for this method. One of the most common examples where this type of systems can appear is a structure controlled by a set of actuators. There are a whole set of control design methods where the control forces that acts on the structures are determined as position feedback or as position and velocity feedback [11–13]. From a matrix algebra point of view the control forces acting on the structure are represented as a linear combination of the displacements and velocities at different points on the structure. The method as it was presented in [6] can only cover the case where the control gain matrix is diagonal as it was studied in [13].

By following some of the solutions presented in [6] this paper extends the method to structural control and gives a new justification of the approximate modification formula that was derived from Sherman–Morrison matrix inversion formula and presents a new method of controlling the approximation accuracy. This approximate formula can then be generalised and it is revealed that it can be used within some limits as an approximate matrix inversion formula. This paper does not give a formal mathematical proof for the formula but rather based on simulation results shows that for a reasonable number of cases the formula is valid and can be used in problems of low frequency vibration response estimation and control.

2. Matrix inversion formulae and their use in estimation and structural dynamics

The Sherman–Morrison–Woodbury matrix identity is used when a matrix $\mathbf{B} \in \mathscr{M}_{n\times n}$, where $\mathscr{M}_{n\times n}$ is the $n \times n$ matrix dimensions space over $\mathbb{R}$ or $\mathbb{C}$ is obtained by a small alteration of a non-singular matrix $\mathbf{A} \in \mathscr{M}_{n\times n}$ and the difference between the two could be written as $\mathbf{A}-\mathbf{B}=\mathbf{U}\mathbf{V}^{\mathrm{T}}$ with $\mathbf{U},\mathbf{V} \in \mathscr{M}_{n\times k}$. Under these conditions the Sherman–Morrison–Woodbury identity states that matrix $\mathbf{A}+\mathbf{U}\mathbf{V}^{\mathrm{T}}$ is also non-singular and its inverse is given by

$$\mathbf{B}^{-1}=(\mathbf{A}+\mathbf{U}\mathbf{V}^{\mathrm{T}})^{-1}=\mathbf{A}^{-1}-\mathbf{A}^{-1}\mathbf{U}(\mathbf{I}_k+\mathbf{V}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{U})^{-1}\mathbf{V}^{\mathrm{T}}\mathbf{A}^{-1} \tag{1}$$

The left-hand side term in the matrix identity possesses the advantage that if k is much smaller than n, the inverse of the modified matrix involves inversion of a reduced dimension matrix with a more advantageous computational cost. In Eq. (1) $\mathbf{I}_k$ is the identity matrix of dimension k which can be written using the elementary k-dimensional vectors $\mathbf{e}_i^k$ as $\mathbf{I}_k=[\mathbf{e}_1^k\mathbf{e}_2^k\ldots\mathbf{e}_k^k]$.

In the special case when $k=1$, the inverse on the right-hand side of (1) becomes a scalar. This simpler formula was presented in different studies and it is usually called the Sherman–Morrison inversion formula. It gives the inverse of the