Mechanical Systems and Signal Processing 37 (2013) 92–104

Contents lists available at SciVerse ScienceDirect

Mechanical Systems and Signal Processing

journal homepage: www.elsevier.com/locate/ymssp

Generalized Fokker–Planck equation with generalized interval probability

Yan Wang *

Woodruff School of Mechanical Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA

ARTICLE INFO

Article history:
Received 11 December 2011
Received in revised form
8 February 2012
Accepted 21 February 2012
Available online 13 March 2012

Keywords:
Generalized interval probability
Generalized interval
Fokker–Planck equation
Chapman–Kolmogorov equation
Path integral

ABSTRACT

The Fokker–Planck equation is widely used to describe the time evolution of stochastic systems in drift-diffusion processes. Yet, it does not differentiate two types of uncertainties: aleatory uncertainty that is inherent randomness and epistemic uncertainty due to lack of perfect knowledge. In this paper, a generalized differential Chapman–Kolmogorov equation based on a new generalized interval probability theory is derived, where epistemic uncertainty is modeled by the generalized interval while the aleatory one is by the probability measure. A generalized Fokker–Planck equation is proposed to describe drift-diffusion processes under both uncertainties. A path integral approach is developed to numerically solve the generalized Fokker–Planck equation. The resulted interval-valued probability density functions rigorously bound the real-valued ones computed from the classical path integral method. The method is demonstrated by numerical examples.

1. Introduction

The Fokker–Planck equation is a general probabilistic approach to describe the dynamics of various stochastic systems, such as physical, chemical, biological and economical ones. It models the time evolution of the probability distribution in a system under uncertainty, which describes generic drift-diffusion processes. However, it does not differentiate the two types of uncertainties. Variability is the inherent randomness in the system because of fluctuation and perturbation. Variability is also referred to as aleatory uncertainty, stochastic uncertainty, simulation uncertainty, and irreducible uncertainty. In contrast, incertitude is due to lack of perfect knowledge or enough information about the system. It is also known as epistemic uncertainty, reducible uncertainty, and model form uncertainty. The classical Fokker–Planck equation models the two types of uncertainties together with one single probability distribution, which only captures the accumulative effect.

The need to separately quantify the two types of uncertainties has been well-recognized (e.g. [1–4]). They need to be represented explicitly if we want to increase the confidence of modeling or simulation results. Neglecting epistemic uncertainty may lead to decisions that are not robust. Sensitivity analysis is a typical way to assess robustness, which is to check how much deviation the analysis result may have if input distribution parameters or distribution types deviate away from the ones used in the analysis. Second-order Monte Carlo sampling can also be applied where samples of model parameters are drawn to assess different models and the effect of epistemic uncertainty can be revealed. Obviously, considerable workload is required for both approaches. Mixing epistemic and aleatory uncertainties may increase costs of risk management. If extra knowledge or information of the collected data is available, they can be further clustered into

* Tel.: +1 404 894 4714; fax: +1 404 894 9342.
E-mail address: yan.wang@me.gatech.edu

0888-3270/$ - see front matter
doi:10.1016/j.ymssp.2012.02.013

smaller groups or intrinsic mathematical relationships can be identified so that variance can be reduced, which reflects pure randomness more accurately for risk analysis.

In studying the dynamics of stochastic systems, it is desirable that aleatory and epistemic uncertainties are separately quantified so that their respective effects can be easily computed, distinguished, and analyzed. In this paper, we propose an efficient approach to quantify aleatory and epistemic uncertainties by interval probability in modeling drift-diffusion processes. Instead of a precise value of probability $P(E)=p$ associated with an event E, a pair of lower and upper probabilities $P(E)=[\underline{p},\overline{p}]$ are used to represent a set of possible values. The range of the interval $[\underline{p},\overline{p}]$ captures the epistemic uncertainty component. Interval probability thus differentiates incertitude from variability both qualitatively and quantitatively in a concise form.

The general purpose of using interval probability or imprecise probability in analyzing system dynamics is to improve the robustness of prediction in a generic and efficient way. In this paper, a generalized differential Chapman–Kolmogorov equation under both uncertainty components based on a generalized interval probability is first derived. The generalized interval probability provides a convenient calculus structure to estimate lower and upper bounds. Then a path integral approach is developed to numerically solve the generalized Fokker–Planck equation, which is a special case of the generalized differential Chapman–Kolmogorov equation. It is also demonstrated that the interval-valued probability density function as the solution of the generalized Fokker–Planck equation by the proposed path integral method rigorously bounds the real-valued one computed from the classical path integral method. Therefore, the generalized Fokker–Planck equation can effectively quantify the epistemic uncertainty associated with parameters and model forms.

In the remainder of the paper, an overview of relevant work in imprecise probability and path integral methods to solve the classical Fokker–Planck equation are given in Section 2. In Section 3, the generalized differential Chapman–Kolmogorov equation is derived. Section 4 describes the proposed path integral approach to solve the generalized Fokker–Planck equation. In Section 5, two numerical examples are used to demonstrate the new approach, which is able to analyze system dynamics under both uncertainty components.

2. Background

2.1. Imprecise probability

Probability theory provides common ground to quantify uncertainty and so far is the most popular approach. It is based on precise values of probability measures or moments. However, precise probability has limitations in representing epistemic uncertainty. The most significant one is that it does not differentiate *total ignorance* from other probability distributions. Total ignorance means that the analyst has zero knowledge about the system under study. Based on the principle of maximum entropy, uniform distributions are usually applied in this case. A problem arises because introducing a uniform or any particular form of a distribution has itself introduced extra information that is not justifiable by the zero knowledge. Different possible values are equally likely in a uniform distribution, which is not guaranteed to be true when we are totally ignorant. The principle of maximum entropy leads to the Bertrand-style paradoxes such as the Van Fraasen's cube factory [5]. Therefore, "Knowing the unknown" as modeled in precise probability does not represent the total ignorance. In contrast, the interval probability $P=[0,1]$ does.

Another limitation of precise probability is representing *indeterminacy* and *inconsistency* in the context of subjective probability. When people have limited ability to determine the precise values of their own subjective probabilities, precise probability does not capture indeterminacy. Therefore Bayesians who insist on subjective probability still do sensitivity analysis. Furthermore, when subjective probabilities from different people are inconsistent, a precise value does not capture a range of opinions or estimations adequately without assuming some consensus on the precise values for a collection of opinions. "Agreeing to disagree" is not the best way to indicate inconsistency.

Imprecise probability $[\underline{p},\overline{p}]$ combines epistemic uncertainty (as an interval) with aleatory uncertainty (as probability measure), which is regarded as a generalization of traditional probability. Gaining more knowledge can reduce the level of imprecision and indeterminacy, i.e. the interval width. When $\underline{p}=\overline{p}$, the degenerated interval probability becomes a traditional precise one. Our proposed approach uses imprecise probabilities to quantify aleatory and epistemic uncertainties separately. Many forms of imprecise probabilities have been developed. For example, the Dempster–Shafer theory [6,7] characterizes evidence with discrete probability masses associated with a power set of values. The theory of coherent lower previsions [1] models uncertainties with the lower and upper previsions with behavioral interpretations. The possibility theory [8] represents uncertainties with Necessity–Possibility pairs. Probability bound analysis [9] captures uncertain information with pairs of lower and upper distribution functions or p-boxes. F-probability [10] incorporates intervals and represents an interval probability as a set of probabilities which maintain the Kolmogorov properties. A random set [11] is a multi-valued mapping from the probability space to the value space. Fuzzy probability [12] considers probability distributions with fuzzy parameters. A cloud [13] is a combination of fuzzy sets, intervals, and probability distributions.

In the applications of interval probability, the interval bounds $\underline{p}$ and $\overline{p}$ can be solicited as the lowest and highest subjective probabilities about a particular event from a domain expert, where probability represents the degree of belief. One expert may hesitate to offer just a precise value of probability. Different experts could have different beliefs. In both cases, the range of probabilities gives the interval bounds. When used in data analysis with frequency interpretation, the