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Reliability analysis of polynomial systems subject to p-box uncertainties

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ABSTRACT

This paper proposes a reliability analysis framework for systems subject to multiple design requirements that depend polynomially on uncertain parameters. The values these polynomials take at a given realization of the uncertain parameters dictate whether that realization is a failure or a success point. In this paper, reliability analysis refers to the estimation or bounding of the probability of failure for a given model of the uncertainty. The probability distributions of the uncertain parameters are presumed to belong to a given probability box (also known as a p-box). This does not give sufficient information to determine the failure probability of such a system exactly, but does limit the range of values it might take. Two techniques for bounding this range are proposed herein. In the first approach, we calculate the p-box of the requirements functions by propagating all the hyper-rectangles defined by the p-box of the uncertain parameters. In the second approach, we find inner bounding sets of the safe and failure domains and search for the elements of the p-box that minimize and maximize the probability of such sets. Iterative refinement of the bounding sets allows tightening arbitrarily closely the offset between the actual failure probability range and the calculated outer bound. In both techniques, bounds of the functions describing the design requirements over hyper-rectangular sets are calculated and iteratively refined by expanding them using Bernstein bases.

1. Introduction

This paper studies the reliability of a system for which a parametric mathematical model is available. The acceptability of the system depends upon its ability to satisfy several design requirements. These requirements, which are represented by a set of polynomial inequality constraints on selected output metrics, depend on the uncertain parameter $\boldsymbol{p}$. The reliability analysis of the system consists of assessing its ability to satisfy all the requirements when the value of $\boldsymbol{p}$ is uncertain. In this paper the uncertainty in $\boldsymbol{p}$ will be modelled as a p-box. This means that the cumulative distribution function (CDF) for $\boldsymbol{p}$ lies within the bounds set by the p-box, and any of the infinitely many CDFs satisfying such bounds can describe the uncertainty in $\boldsymbol{p}$. A reliability analysis of the system for all members of the p-box yields a range of failure probabilities. The main goal of this paper is the generation of tight outer bounds to this range.

Two conceptually different techniques for bounding the range of failure probabilities are proposed. The first approach constructs the p-box of the output by propagating all the hyper-rectangular sets resulting from a p-box in the inputs.

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This is carried out by making use of the range bounding property of the Bernstein polynomials [1]. In the second approach we search for the elements of the p-box that minimize and maximize the probability of inner and outer bounding sets of the failure domain. These elements will be called *extreme-case distributions* hereafter. Tests for evaluating set containment conditions based on Bernstein polynomials are used to construct such sets. The failure probabilities corresponding to the extreme-case distributions are the limits of the desired bounding intervals.

This paper is organized as follows. Basic concepts from reliability analysis and imprecise probabilities are introduced in Section 2. Section 3 presents strategies for reliability analysis based on interval propagation. Formulations for identifying the extreme-case distributions are presented in Section 4. Finally, a few concluding remarks close the paper.

2. Background

2.1. Basic concepts and notions

A probabilistic uncertainty model of $\boldsymbol{p} \in \mathbb{R}^s$ is prescribed by a random vector supported in the *support set* $\Delta \subseteq \mathbb{R}^s$. This set is composed of all possible uncertain parameter realizations that may occur. This model is fully prescribed by the joint cumulative distribution function $F_{\boldsymbol{p}}(\boldsymbol{p}) : \Delta \to [0,1]$.

Consider a system that depends on the uncertain parameter $\boldsymbol{p}$. The requirements imposed upon such a system are prescribed as the vector[1] inequality, $\boldsymbol{g}(\boldsymbol{p}) < \boldsymbol{0}$. The requirement function $\boldsymbol{g}$ is defined as $\boldsymbol{g} : \mathcal{D} \to \mathbb{R}^{\nu}$, and $\Delta \subseteq \mathcal{D}$, where $\mathcal{D}$ denotes the *master domain*.

The *failure domain*, denoted as $\mathcal{F} \subset \mathbb{R}^s$, is composed of the parameter realizations that fail to satisfy at least one of the requirements. Specifically, the failure domain is given by

$$\mathcal{F} = \bigcup_{i=1}^{\nu} \{\boldsymbol{p} : \boldsymbol{g}_i(\boldsymbol{p}) \ge 0\} = \{\boldsymbol{p} : w(\boldsymbol{p}) \ge 0\}, \tag{1}$$

where $w(\boldsymbol{p}) = \max_{1 \le i \le \nu}\{\boldsymbol{g}_i(\boldsymbol{p})\}$ is the *worst-case requirement function*. The *safe domain*, given by $\mathcal{S} = C(\mathcal{F})$, where $C(\cdot)$ denotes the *complement* set operator $C(\mathcal{Z}) \triangleq \mathcal{D} \backslash \mathcal{Z}$, consists of the parameter realizations satisfying all requirements. The failure probability associated with the distribution function $F_{\boldsymbol{p}}$ is given by

$$P_{F_{\boldsymbol{p}}}[\mathcal{F}] = \int_{\mathcal{F}} 1 \, dF_{\boldsymbol{p}}, \tag{2}$$

where $P_{F_{\boldsymbol{p}}}[\cdot]$ is the probability operator determined by the distribution $F_{\boldsymbol{p}}$.

Techniques for bounding $\mathcal{F}$ and $\mathcal{S}$ will be presented below. The resulting bounding sets are unions of hyper-rectangles. The *hyper-rectangle* having $\boldsymbol{m} > \boldsymbol{0}$ as the vector of half-lengths of the sides and $\overline{\boldsymbol{p}}$ as its geometric center is given by

$$\mathcal{R}(\overline{\boldsymbol{p}},\boldsymbol{m}) = \{\boldsymbol{p} : \overline{\boldsymbol{p}} - \boldsymbol{m} < \boldsymbol{p} \le \overline{\boldsymbol{p}} + \boldsymbol{m}\} = \boldsymbol{\delta}(\overline{\boldsymbol{p}} - \boldsymbol{m}, \overline{\boldsymbol{p}} + \boldsymbol{m}), \tag{3}$$

where

$$\boldsymbol{\delta}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{\delta}_1 \times \boldsymbol{\delta}_2 \times \cdots \times \boldsymbol{\delta}_s \tag{4}$$

is the Cartesian product of the intervals $\boldsymbol{\delta}_i = (\boldsymbol{x}_i, \boldsymbol{y}_i]$. Hyper-rectangles can be subdivided into smaller hyper-rectangles without overlap, so that each point of the original hyper-rectangle falls into exactly one of the subdividing hyper-rectangles. Under these circumstances, the larger hyper-rectangle is said to have been *partitioned* or *subdivided* into smaller hyper-rectangles. If $\rho(\mathcal{R}) = \{\mathcal{R}_1, \ldots, \mathcal{R}_t\}$ is a pairwise disjoint collection of hyper-rectangles, where $\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_j$, then ρ is a partition of $\mathcal{R}$. Multiple subdivision schemes are possible. For instance, a bisection-based subdivision of $\mathcal{R}$ in the ith direction, $i \le s$, is given by

$$\rho(\mathcal{R}) = \{\mathcal{R}(\overline{\boldsymbol{p}} + \boldsymbol{w}, \boldsymbol{m} - \boldsymbol{w}), \mathcal{R}(\overline{\boldsymbol{p}} - \boldsymbol{w}, \boldsymbol{m} - \boldsymbol{w})\}, \tag{5}$$

where $\boldsymbol{w} = [0, \ldots, 0, \boldsymbol{m}_i/2, 0, \ldots, 0]$. Throughout this paper the input to ρ is either a single hyper-rectangle or the union of several of them. In the latter case, each sub-rectangle comprising the input set will be bisected using Eq. (5).

2.2. Bernstein polynomials

The image of a hyper-rectangle when mapped by a multivariable polynomial is a bounded interval. By expanding that polynomial using a Bernstein basis over that rectangle, rigorous bounds to such an interval can be calculated using mere algebraic manipulations. A high-level description of the tasks to be performed using this technique is presented next. The mathematical background and algorithmic implementation of these tasks is presented in detail in the Appendix. Bernstein polynomials will be used for:

1. Bounding the range of the multivariable piecewise polynomial $w(\boldsymbol{p})$ for all $\boldsymbol{p} \in \mathcal{R} \subset \mathbb{R}^s$ (see Eq. (A.20)).

[1] Throughout this paper, it is assumed that vector inequalities hold component-wise, super-indices denote a particular vector or set, and sub-indices refer to vector components; e.g., $\boldsymbol{p}_i^{(j)}$ is the ith component of the vector $\boldsymbol{p}^{(j)}$.