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Similarity of signal processing effect between Hankel matrix-based SVD and wavelet transform and its mechanism analysis

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It is pointed out that signal processing effect of singular value decomposition (SVD) is very similar to that of wavelet transform when Hankel matrix is used. It is proved that a signal can be decomposed into the linear sum of a series of component signals by Hankel matrix-based SVD, and essentially what the component signals reflect are projections of original signal on the orthonormal bases of m-dimensional and n-dimensional vector spaces. The similarity mechanism of signal processing between SVD and wavelet transform is analyzed from the angle of basis of vector space and characteristic of Hankel matrix. The orthogonality of the component signals got by SVD and wavelet transform is also studied. It is discovered that singularity of signal can also be detected by Hankel matrix-based SVD, and compared with wavelet transform, there are two characteristics in SVD for singularity detection, one is that the order of vanishing moment of SVD component signals is increased progressively and the one of the nth SVD component signal is '$n-1$', so singular points with different Lip index can all be detected, the other is that the width of impulse indicating the position of singularity will always keep the same throughout all SVD components and this width is determined by the column number of Hankel matrix.

1. Introduction

The definition of singular value decomposition (SVD) is: for a matrix $\boldsymbol{A}\in\mathbf{R}^{m\times n}$, two orthogonal matrices $\boldsymbol{U}=[\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m]\in\mathbf{R}^{m\times m}$ and $\boldsymbol{V}=[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n]\in\mathbf{R}^{n\times n}$ are surely existed to meet the following equation [1]

$$\boldsymbol{A}=\boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^T \tag{1}$$

where $\boldsymbol{S}=[diag(\sigma_1, \sigma_2, \ldots, \sigma_q), \boldsymbol{O}]$ or its transposition, which is decided by $m<n$ or $m>n$, $\boldsymbol{S}\in\mathbf{R}^{m\times n}$, while $\boldsymbol{O}$ is zero matrix, $q=\min(m, n)$, and $\sigma_1\geqslant\sigma_2\geqslant\cdots\geqslant\sigma_q>0$. These σ_i ($i=1, 2, \ldots, q$) are called the singular values of matrix $\boldsymbol{A}$.

SVD and wavelet transform are two different theories that can never be mentioned in the same breath and their mathematical foundations are completely different from each other, but we discovered that if suitable matrix $\boldsymbol{A}$ is adopted, SVD will show the very surprisingly similar effect of signal processing to that of wavelet transform. Many kinds of matrices can be created by one-dimensional signal, such as Toeplitz matrix, cycle matrix and Hankel matrix, or matrix can be created by continuous interception of signal. The difference is the creation way of matrix, the difference will be the effect of signal processing of SVD [2]. To our surprise, SVD can achieve the very similar effect of signal processing to wavelet transform when $\boldsymbol{A}$ is Hankel matrix, such as that the original signal can be decomposed into the combination of a series of detail

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signals and approximation signal by both methods, and the singular points in signal can be detected by both methods. Furthermore, in some aspects, the decomposition results of SVD have the special advantage of those of the wavelet transform. Why the signal processing effect of these two completely different theories are so similar to each other? This question is worth studying.

To explain this phenomenon reasonably, the essence of signal decomposition of these two methods should be ascertained. As is well known, in fact the essence of wavelet transform is to use two high and low pass filters to decompose a signal into a series of component signals locating in different frequency bands, while as for SVD, there is no such direct explanation. Most references on the signal processing of SVD always place their emphasis on practicality and SVD is only used as a tool to solve certain specific signal processing problem, such as feature extraction [3–6], data compression [7], noise reduction [8–11], speech coding [12] and so on, while the study on the essence of signal decomposition of SVD is very few, in this paper this essence has been ascertained, and on this basis the similarity mechanism of signal processing between SVD and wavelet transform is explained from the angle of basis of vector space. Besides, the orthogonality of component signals, got by these two methods, is also studied and compared theoretically.

On the basis of theory analysis, the practically similar effect of signal processing of these two methods are demonstrated by several examples, and it is pointed out that for structure characteristic of Hankel matrix in itself, the first SVD component signal can correspond to the approximation signal of the last scale in wavelet transform, while the other SVD component signals can correspond to the detail signals in the wavelet transform.

On the other hand, some differences between these two methods are also studied, such as that there is no phase shift in the decomposition results of SVD, while phase lag is existed in those of wavelet transform. And as to the singularity detection, for that the order of vanishing moment of each kind of wavelet is fixed, once wavelet is selected, the singular index that can be detected by wavelet transform will also be fixed, and, moreover, the width of impulse indicating the position of singular point will become broad and broad with the increase of scale. While SVD is different from this, it is discovered that the order of vanishing moment of SVD components is increased progressively and the one of the nth SVD component signal is '$n-1$', so singular points with different singular index can also be detected by different SVD components, besides, the width of impulse indicating the position of singular point will always keep the same throughout all SVD components and it is found that this width is determined by the column number of Hankel matrix, so in these aspects, SVD has the advantage of wavelet transform.

2. Essence of signal decomposition of Hankel matrix-based SVD

For a discrete signal $\boldsymbol{X}=[x(1), x(2), \ldots, x(N)]$, Hankel matrix can be created by this signal as follows:

$$\boldsymbol{A}=\begin{bmatrix} x(1) & x(2) & \cdots & x(n) \\ x(2) & x(3) & \cdots & x(n+1) \\ \vdots & \vdots & \vdots & \vdots \\ x(N-n+1) & x(N-n+2) & \cdots & x(N) \end{bmatrix}$$

where $1<n<N$, let $m=N-n+1$, then $\boldsymbol{A}\in\mathbf{R}^{m\times n}$.

In order to realize the isolation of signal using SVD, the Eq. (1) should be converted to the form of column vectors $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$

$$\boldsymbol{A}=\sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^T+\sigma_2\boldsymbol{u}_2\boldsymbol{v}_2^T+\cdots+\sigma_q\boldsymbol{u}_q\boldsymbol{v}_q^T \tag{2}$$

where $\boldsymbol{u}_i\in\mathbf{R}^{m\times 1}$, $\boldsymbol{v}_i\in\mathbf{R}^{n\times 1}$, $i=1, 2, \ldots, q$, $q=\min(m, n)$. According to the SVD theory, the vectors $\boldsymbol{u}_i$ are orthonormal to one another and they form the orthonormal bases of m-dimensional space; the vectors $\boldsymbol{v}_i$ are also orthonormal to one another and they form the orthonormal bases of n-dimensional space [1].

Let $\boldsymbol{A}_i=\sigma_i\boldsymbol{u}_i\boldsymbol{v}_i^T$, then $\boldsymbol{A}_i\in\mathbf{R}^{m\times n}$ also. Supposing that $\boldsymbol{P}_{i,1}$ is the first row vector of $\boldsymbol{A}_i$, and $\boldsymbol{H}_{i,n}$ is a column vector in the last column of $\boldsymbol{A}_i$, which is shown in Fig. 1, according to the creation principle of Hankel matrix, if $\boldsymbol{P}_{i,1}$ and the transposition of $\boldsymbol{H}_{i,n}$ are linked together, then a SVD component signal $\boldsymbol{P}_i$ can be obtained, which can be expressed as the vector form

$$\boldsymbol{P}_i=(\boldsymbol{P}_{i,1}, \boldsymbol{H}_{i,n}^T).\quad \boldsymbol{P}_{i,1}\in\boldsymbol{R}^{1\times n},\quad \boldsymbol{H}_{i,n}\in\boldsymbol{R}^{(m-1)\times 1} \tag{3}$$

$$\boldsymbol{A}_i=\begin{bmatrix} x_i(1) & x_i(2) & \cdots & x_i(n) \\ x_i(2) & x_i(3) & \cdots & x_i(n+1) \\ \vdots & \vdots & \vdots & \vdots \\ x_i(N-n+1) & x_i(N-n+2) & \cdots & x_i(N) \end{bmatrix}\begin{matrix} P_{i,1} \\ \\ \\ \\ \end{matrix}$$

$H_{i,n}$

Fig. 1. The forming principle of component signal $\boldsymbol{P}_i$ when Hankel matrix is used.