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Fast communication

On the steady-state mean squared error of the fixed-point LMS algorithm

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Abstract

This communication studies the quantization effects on the steady-state performance of a fixed-point implementation of the Least Mean Squares (LMS) adaptive algorithm. Based on experimental observations, we introduce a new *intermediate* mode of operation and develop a simplified theoretical approach to explain the behaviour caused by quantization effects in this mode. We also review the *stall* mode and provide a new expression that predicts the discontinuous behaviour of the steady-state mean squared error as a function of the input signal power. Combined with a previous analysis of quantization effects in *stochastic gradient* mode, this study provides analytical expressions for the steady-state mean squared error for the full range of step-size values. We present experimental results that are in a good agreement with theoretical predictions to validate our model.

Keywords: LMS; Fixed-point arithmetic; Quantization effects; Adaptive filtering

1. Introduction

The Least Mean Squares (LMS) algorithm [1] is widely used in adaptive filtering. This algorithm recursively updates the vector of coefficients $\mathbf{w}(k) = [w_0(k), w_1(k), \ldots, w_{N-1}(k)]^t$ of an FIR filter according to the following equations

$$e(k) = y(k) - \mathbf{w}^t(k)\mathbf{x}(k), \tag{1a}$$

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \mu e(k)\mathbf{x}(k), \tag{1b}$$

where k is the discrete time index, $e(k)$ is the estimation error, $\mathbf{x}(k) = [x(k), x(k-1), \ldots, x(k-N+1)]^t$ is the input signal vector, $y(k)$ is the reference signal to be estimated, and μ is an adaptation parameter. The speed of convergence of the algorithm towards the optimal Wiener solution, as well as the power of the residual error after convergence (i.e., in steady-state) depend on μ. These properties of the LMS are well established for infinite precision arithmetic (see [2] and references therein).

When LMS is implemented on a fixed-point processor, quantization errors affect its performance and the mean squared error (MSE), i.e. $E\{|e(k)|^2\}$ where $E\{\cdot\}$ denotes statistical expectation, may be significantly higher than the one expected in infinite precision. Analysis of the quantization effects on LMS performance goes back to the work of Gitlin et al. [3] who studied the variations of the MSE as a function of the step-size in a digitally implemented LMS. They reported two main

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observations: (a) the MSE after the algorithm converges is much higher than the one expected from a quantization of the algorithm variables and (b) due to quantization effects, the adaptation may stop and in this case, the MSE may be actually reduced by increasing the step-size. Caraiscos and Liu [4] presented an analysis of quantization errors in steady-state for fixed-point and floating-point arithmetic. In their analysis, they modelled quantization errors as white noise and obtained an analytical expression for the residual error. However, the error model they used is only valid when the adaptation is not stopped by quantization effects—in this situation, the quantization has a low impact on the steady-state MSE. Alexander [5] used the same white noise model for quantization errors to analyse the behaviour of the finite precision LMS algorithm in the transient regime.

Bermudez and Bershad [6] recognized the drawback of the error model used in [4] and [5] and its non-validity in a situation where adaptation is stopped by quantization effects. They proposed a non-linear analytical model for the quantization function. By using a conditional moment technique, for a white Gaussian input and a small adaptation step size, they derived recursive equations which can be numerically solved to give the MSE in both transient and steady-state regimes [6]. They investigated the steady-state behaviour of the quantized LMS algorithm for small step-size and showed that the stalling behaviour is indeed a "slow-down" phenomenon. Under limiting assumptions, their model predicts a steady-state MSE that is nearly independent on the number of bits [7]. In our study, we observed the steady-state behaviour of the LMS for all range of values of the adaptation step-size and noticed a clear difference between our simulation results and the MSE predicted by the model in [7], as we will show in Section 3.

The studies cited above explained and modelled the LMS behaviour in different conditions but did not provide an analytical expression of the steady-state MSE (SS-MSE) for all these conditions. In particular, the intermediate region between the *stall* mode, where adaptation is stopped, and the *stochastic gradient* mode, where the analysis of [4] is applicable, has not been previously investigated. In this work, we introduce a new *intermediate* mode to characterize the algorithm behaviour in this region and develop a simplified theoretical model that provides an analytic expression of the corresponding SS-MSE values for a white stationary Gaussian input signal. We also review the stall mode and provide a new expression that predicts the discontinuous behaviour of the SS-MSE as a function of the input signal power. Combined with the analysis of quantization effects in stochastic gradient mode, this study provides analytical expressions for the SS-MSE for the complete range of step-size parameter values. In particular, the value of step-size corresponding to the onset of the stall mode can be predicted accurately, so that stalling can be avoided by judiciously choosing the step size value.

The outline of the paper is as follows. In Section 2 we present the theoretical analysis and developments leading to analytical expressions for the SS-MSE for different operating conditions of the algorithm. In Section 3 we present experimental results. A brief conclusion follows in Section 4.

2. Theoretical analysis

In our analysis of the finite precision LMS algorithm we assume that the input signal $x(k)$ is a white stationary Gaussian process with zero mean and variance σ_x^2, and the reference signal $y(k)$ is written as

$$y(k) = \mathbf{w}_o^t \mathbf{x}(k) + n(k), \tag{2}$$

where $\mathbf{w}_o$ is the optimal vector of coefficients and $n(k)$ is a white stationary Gaussian noise independent of $x(k)$, with zero mean and variance σ_n^2. We assume that signals and filter coefficients are real-valued but this analysis can be easily generalized to the complex case.

Under the above assumption on the input signal $x(k)$, the LMS algorithm converges in the mean square for values of μ in the range [8]

$$0 < \mu < 2/(N\sigma_x^2), \tag{3}$$

where N is assumed to be large.[1] The SS-MSE, i.e. after the algorithm converges, is given by [4]

$$\xi = \lim_{k\to\infty} E\{|e(k)|^2\} = \frac{\xi_{\min}}{1-\mu N\sigma_x^2/2}, \tag{4}$$

where $\xi_{\min}$ is the MSE of the optimum filter, equal to σ_n^2 if the adaptive filter is long enough to cover the impulse response to be estimated.

[1] Variations of this result can be found in the literature (see [2] and references therein). However, for the model under consideration here (i.e. white stationary Gaussian processes and large N), the values predicted by (3) are sufficiently accurate and in good agreement with experimental observations.