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Semi-widely linear estimation of $\mathbb{C}^\eta$-proper quaternion random signal vectors under Gaussian and stationary conditions

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ABSTRACT

Recursive algorithms for estimating quaternion signal vectors are provided. Assuming stationarity, Gaussianity and $\mathbb{C}^\eta$-properness, two estimation problems are treated: the prediction and fixed-point smoothing problems. The problems are formulated in a very general way and solved following a semi-widely linear processing. The suggested solutions are derived taking additionally into account the information supplied by a *square* version of the quaternion observations. This extra information makes it possible to give improved estimators which outperform the corresponding full-widely linear estimators that ignore such information. A numerical example pertaining to the nonlinear prediction and fixed-point smoothing problems illustrates the application of the algorithms.

1. Introduction

Quaternions are hyper-complex numbers which allow for convenient and effective statistical modeling of multichannel signals. They have received much attention in recent years. For example, quaternion signal processing has encountered applications in wind forecasting [1], aerospace [2], computer graphics problems [3], image processing [4], vector sensor [5], processing of polarized waves [6], and design of space-time block codes [7].

The adequate type of linear processing in the quaternion domain depends on the kind of quaternion properness [8]. Unlike the case of complex vectors, there exist three different kinds of quaternion properness, which are based on the vanishing of three different complementary covariance matrices. In general, the optimal linear processing is full-widely linear, which means that we must simultaneously operate on the quaternion vector and its three involutions. However, in the case of jointly $\mathbb{Q}$-proper or $\mathbb{C}^\eta$-proper vectors, which constitute the two principal types of properness, the optimal processing reduces to the conventional or semi-widely linear processing, respectively. Conventional processing ignores the vector involutions while semi-widely linear processing makes use of the quaternion vector and its involution over the pure unit quaternion η.

Algorithms adapted for improper signals can fail or suffer from slow convergence when they are used for proper signals [9]. Thus, it is essential to devise particular algorithms for proper signals. In fact, a variety of problems have been solved using the properness hypothesis, e.g., classification of polarized signals [6], detection [10,11], quaternion VAR modeling and estimation [12], etc. On the other hand, several methods are available to determine whether a quaternion random vector is $\mathbb{Q}$-proper, $\mathbb{C}^\eta$-proper, or improper [13–15].

An important problem in statistical signal processing is the estimation of a signal from the information supplied

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by another signal. As is well-known, the optimal estimator under the minimum mean-squared error (MMSE) criterion is the conditional expectation. Moreover, the Gaussian hypothesis has been widely used in this problem. For instance, the Kalman filter is the optimal estimation algorithm when the signal model is assumed linear and both state and observation noise are additive Gaussian (see, e.g., [16]). In the nonlinear case, the optimal filtering problem poses a challenge since it entails maintaining the complete description of the conditional probability density function which in general requires an infinite number of parameters. So, the nonlinear problem has been addressed by using approximations in order to develop suboptimal estimators [16–19]. Notable examples which apply Gaussian approximations are the extended Kalman filter [16], the unscented Kalman filter [20], the Gauss-Hermite filter [21], the central difference filter [21] and the Gaussian sum filters [22,23].

From this background, this paper concentrates on the nonlinear estimation of quaternion random signal vectors with unknown probability distributions and under the Gaussianity and $\mathbb{C}^{\eta}$-properness hypotheses of the observations. Specifically, two minimum MMSE estimation problems are tackled following a semi-widely linear approach: the linear prediction and linear fixed-point smoothing problems. The suggested semi-widely linear solutions improve the corresponding conventional solutions in two directions: they offer a better performance (in MSE sense) and they involve matrices with lower dimensions. The former advantage is achieved by considering the information supplied by a *square* version of the quaternion observations and the latter by the $\mathbb{C}^{\eta}$-properness and Gaussianity. Recursivity is another desirable characteristic of the solutions and it will be attained by assuming stationarity conditions. Likewise, the formulation of the estimation problem considered is very general and the technique is easily adapted to a wide range of applications. As we will see in Section 6, the algorithms can be employed, for example, to estimate a nonlinear function of the signal of interest.

As noted above, the improved estimators are obtained by incorporating the information furnished by the *square* observations. For that, it is necessary to introduce the concept of *square* quaternion vector. Moreover, we show that $\mathbb{C}^{\eta}$-properness and Gaussianity ensure the $\mathbb{C}^{\eta}$-properness of the *square* quaternion vectors and the lack of correlation between the augmented quaternion vector and its augmented *square* vector (see Lemma 1). These are key properties that allow us to split up the semi-widely linear estimators as a sum of two terms: the conventional quaternion widely linear (QWL) estimator which ignores the *square* quaternion vector and a second estimator built from this last quaternion vector. Such a representation of the suggested estimators makes them improved versions of the conventional estimators.

The paper is organized as follows. In Section 2 we firstly provide some basic definitions and introduce notation, along with the QWL processing. In Section 3 the $\mathbb{C}^{\eta}$-properness is studied for quaternion signal vectors and its impact on *square* quaternion signal vectors is investigated. Section 4 addresses the semi-widely linear prediction problem. Firstly, we formulate the general prediction problem and derive the optimum predictor under a QWL processing. Then we review the one-stage prediction problem under the $\mathbb{C}^{\eta}$-properness and stationarity conditions, and a version of the Durbin–Levinson algorithm is presented. Finally, we study the general prediction problem by additionally assuming Gaussianity and incorporating the information supplied by the *square* vector of the observation quaternion process. Section 5 treats the semi-widely linear fixed-point smoothing problem. Section 6 considers a simulation example which demonstrates the practical application of the algorithms and proves experimentally the superiority of the suggested solutions in relation to the one derived following a QWL processing. Section 7 provides our concluding comments. To preserve continuity in our presentation, all proofs are deferred to an Appendix.

2. Preliminaries

Throughout this paper, all the random variables are assumed to have zero-mean. Next we introduce the basic notation. We use boldfaced upper case letters to denote matrices ($\mathbf{A}$), boldfaced lower case letters for column vectors ($\mathbf{a}$), and lightfaced lower case letters for scalar quantities (a). Superscripts $*$, T and H represent the quaternion (or complex) conjugate, transpose and Hermitian, respectively. $\mathbf{0}_{n\times m}$ denotes the $n\times m$ zero matrix and $\mathbf{I}_m$ an identity matrix of dimension m. The real and imaginary parts of a complex number will be denoted by $\mathcal{R}\{\cdot\}$ and $\mathcal{I}\{\cdot\}$, respectively. The notation $\mathbf{A}\in\mathbb{R}^{n\times m}$ (respectively $\mathbf{A}\in\mathbb{C}^{n\times m}$ or $\mathbf{A}\in\mathbb{H}^{n\times m}$) means that $\mathbf{A}$ is a real (respectively complex or quaternion) $n\times m$ matrix. We will remove a dimension in the case of vectors (e.g., $\mathbf{r}\in\mathbb{H}^m$ means that $\mathbf{r}$ is an m-dimensional quaternion vector).

If $\boldsymbol{\alpha}=[\alpha_1,\dots,\alpha_m]^T\in\mathbb{C}^m$ then the notation $\boldsymbol{\alpha}^{(2)}$ stands for the vector with square components, i.e. $\boldsymbol{\alpha}^{(2)}=[\alpha_1^2,\dots,\alpha_m^2]^T$. Analogously, if $\boldsymbol{\Omega}=\{\omega_{ij}\}\in\mathbb{C}^{m\times q}$ then $\boldsymbol{\Omega}^{(2)}$ denotes an $m\times q$ matrix of entries ω_{ij}^2. Finally, E is the expectation operator, $P_{\mathcal{K}}$ is the projection operator on an arbitrary set $\mathcal{K}$, $\mathcal{K}_1\oplus\mathcal{K}_2$ is the direct sum of the sets $\mathcal{K}_1$ and $\mathcal{K}_2$ and $\otimes$ is the Kronecker product.

Consider the quaternion random signal

$$r(t)=a(t)+\mathrm{i}b(t)+\mathrm{j}c(t)+\mathrm{k}d(t) \tag{1}$$

where $a(t)$, $b(t)$, $c(t)$ and $d(t)$ are real random signals and the imaginary units $(\mathrm{i},\mathrm{j},\mathrm{k})$ satisfy

$$\begin{aligned}&\mathrm{i}^2=\mathrm{j}^2=\mathrm{k}^2=\mathrm{ijk}=-1\\&\mathrm{ij}=\mathrm{k}=-\mathrm{ji}\\&\mathrm{jk}=\mathrm{i}=-\mathrm{kj}\\&\mathrm{ki}=\mathrm{j}=-\mathrm{ik}\end{aligned}$$

The representation in (1) can be generalized to orthogonal bases of the form [8]:

$$\begin{bmatrix}1\\ \eta\\ \eta'\\ \eta''\end{bmatrix}=\begin{bmatrix}1 & \mathbf{0}_{1\times 3}\\ \mathbf{0}_{3\times 1} & \mathbf{A}\end{bmatrix}\begin{bmatrix}1\\ \mathrm{i}\\ \mathrm{j}\\ \mathrm{k}\end{bmatrix}$$

where $\mathbf{A}\in\mathbb{R}^{3\times 3}$ is an orthogonal matrix (i.e., $\mathbf{A}^T\mathbf{A}=\mathbf{I}_3$). Likewise, it is assumed that the signs of the rows of $\mathbf{A}$ are