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An algebraic fractional order differentiator for a class of signals satisfying a linear differential equation

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ABSTRACT

This paper aims at designing a digital fractional order differentiator for a class of signals satisfying a linear differential equation to estimate fractional derivatives with an arbitrary order in noisy case, where the input can be unknown or known with noises. Firstly, an integer order differentiator for the input is constructed using a truncated Jacobi orthogonal series expansion. Then, a new algebraic formula for the Riemann–Liouville derivative is derived, which is enlightened by the algebraic parametric method. Secondly, a digital fractional order differentiator is proposed using a numerical integration method in discrete noisy case. Then, the noise error contribution is analyzed, where an error bound useful for the selection of the design parameter is provided. Finally, numerical examples illustrate the accuracy and the robustness of the proposed fractional order differentiator.

1. Introduction

Fractional calculus has a long history and has been becoming very useful in many scientific and engineering fields, including control, flow propagation, signal processing, and electrical networks (see, e.g. [1–8]). An interesting research topic on fractional calculus is related to the estimation of the fractional order derivatives of an unknown signal from its discrete noisy observation. The objective is to design digital fractional order differentiators, which should be robust against noises. Various robust fractional order differentiators have been proposed in the frequency domain (see, e.g. [9,10]) and in the time domain (see, e.g. [11–16]). They can be divided into two classes: fractional order model-free differentiators (see, e.g. [9–15]) and fractional order model-based differentiators (see, e.g. [16]). The first class of fractional order differentiators are obtained by truncating the analytical expression. Hence, this generates truncated errors even in noise-free case (see, e.g. [13]). The second class of fractional order differentiators are obtained from the differential equations of considered signals. They do not introduce any truncated errors.

Existing fractional order differentiators are usually extensions of integer order differentiators. Among the exising methods, the recent algebraic parametric method originally introduced by Fliess and Sira-Ramírez for linear identification [17] has been applied to design integer order model-free differentiators (see, e.g. [18–23]), and integer order model-based differentiators (see, e.g. [24–26]). The idea of this method is to apply some algebraic operations (such as differentiations and multiplications), in the frequency domain, to the equation of the studied signal. The obtained differentiators are exactly given by algebraic integral formulae in the time domain. It has been shown in [27,28] that, thanks to the integral formulae, these differentiators exhibit

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good robustness properties with respect to corrupting noises even if the statistical properties of the noises are unknown. Very recently, the algebraic integer order model-free differentiators have been extended to fractional case [12,13,29]. However, the algebraic parametric method has not been applied for fractional order model-based differentiators.

The modulating functions method is another method which has been extended to the fractional case. This method has been introduced by Shinbrot [30]. It gives similar results to the algebraic parametric method but works in the time domain. In [16], generalized modulating functions have been introduced to design fractional order model-based differentiators. However, the generalized modulating functions are more complex to construct than the classical ones.

The aim of this paper is to apply the algebraic parametric method to design a robust fractional order model-based differentiator. Moreover, it will be shown that the proposed differentiator can also be obtained by classical modulating functions without using generalized modulating functions. For this purpose, we will focus on a specific class of signals satisfying a linear differential equation, where the input can be unknown or known with noises.

This paper is organized as follows: definitions and some useful properties of fractional calculus, modulating functions, and Jacobi orthogonal polynomials are recalled in Section 2. The main results are given in Section 3. Firstly, an integer order differentiator for the input is constructed using a truncated Jacobi orthogonal series expansion. Secondly, the algebraic parametric method is applied to express the Riemann–Liouville integrals and derivatives of the considered signal by algebraic formulae in continuous noise-free case. Then, it is shown that these integral formulae can also be obtained using the modulating functions method. Thirdly, a digital fractional order differentiator is introduced in discrete noisy case. Moreover, some error analysis is given. In Section 4, numerical results illustrate the accuracy and the robustness of the proposed fractional order differentiator. Finally, conclusions are outlined in Section 5.

2. Preliminary

2.1. Problem formulation

In this paper, a class of signals satisfying the following differential equation are considered:

$$\forall t \in I, \quad \sum_{i=0}^{n} a_i y^{(i)}(t) = u(t), \tag{1}$$

where $n \in \mathbb{N}^*$, $a_n \in \mathbb{R}^*$, $a_i \in \mathbb{R}$, for $i = 0, \ldots, n-1$, $y \in \mathcal{C}^n(I)$, $u \in \mathcal{C}^n(I)$, and $I = [0, h] \subset \mathbb{R}_+$. If (1) is considered as a linear system with the input u, then y is the corresponding output. The objective of this paper is to estimate the Riemann–Liouville fractional derivatives of the output y in noisy environment, where the input can be unknown or known with noises. For this purpose, some useful tools are recalled in the following subsections.

2.2. Riemann–Liouville integrals and derivatives

Definition 1 (*Diethelm [5, p. 13]*). Let $\beta \in \mathbb{R}_+^*$, and f be a continuous function defined on $\mathbb{R}$. Then, the βth order Riemann–Liouville fractional integral of f is defined by $\forall t \in \mathbb{R}_+^*$,

$$\mathrm{J}_t^{\beta} f(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau)\, d\tau, \tag{2}$$

where $\Gamma(\cdot)$ is the Gamma function defined by $\Gamma(z) = \int_0^{\infty} \exp(-x) x^{z-1}\, dx$ and satisfies $\Gamma(z+1) = z\Gamma(z)$ (see [31, pp. 255–256]).

Definition 2 (*Podlubny [4, p. 62]*). Let $\alpha \in \mathbb{R}_+^*$ with $l-1 \le \alpha < l$, $l \in \mathbb{N}^*$, and $f \in \mathcal{C}^l(\mathbb{R})$, where $\mathcal{C}^l(\mathbb{R})$ refers to the set of functions being l-times continuously differentiable on $\mathbb{R}$. Then, the αth order Riemann–Liouville fractional derivative of f is defined as follows: $\forall t \in \mathbb{R}_+^*$,

$$\mathrm{D}_t^{\alpha} f(t) := \frac{d^l}{dt^l} \{\mathrm{J}_t^{l-\alpha} f(t)\} = \frac{1}{\Gamma(l-\alpha)} \frac{d^l}{dt^l} \int_0^t (t-\tau)^{l-\alpha-1} f(\tau)\, d\tau. \tag{3}$$

Remark that if $0 < \beta < 1$, then the integral given in (2) is improper. Hence, the βth order Riemann–Liouville integral is defined by an improper integral. Moreover, according to (3)Riemann–Liouville derivatives are also defined by improper integrals, which are the integer order derivatives of the Riemann–Liouville integrals of order smaller than 1.

According to (2) and (3), Riemann–Liouville integrals and derivatives satisfy the following additive index laws.

- Let $\beta \in \mathbb{R}_+ / \mathbb{N}$, $n \in \mathbb{N}$, and $f \in \mathcal{C}^n(\mathbb{R})$. Then, we have [4, p. 71] $\forall t \in \mathbb{R}_+^*$,

$$\frac{d^n}{dt^n} \left\{ \mathrm{J}_t^{\beta} f(t) \right\} = \begin{cases} \mathrm{J}_t^{\beta-n} f(t) & \text{if } \beta > n, \\ \mathrm{D}_t^{n-\beta} f(t) & \text{else.} \end{cases} \tag{4}$$

- Let $\alpha \in \mathbb{R}_+^*$ with $l-1 \le \alpha < l$, $l \in \mathbb{N}^*$, and $f \in \mathcal{C}^{l+n}(\mathbb{R})$. Then, we have $\forall t \in \mathbb{R}_+^*$,

$$\frac{d^n}{dt^n} \{\mathrm{D}_t^{\alpha} f(t)\} = \mathrm{D}_t^{\alpha+n} f(t). \tag{5}$$

In the following theorem, the Leibniz formula for Riemann–Liouville integrals is recalled. The Leibniz formula for Riemann–Liouville derivatives can be found in [5, p. 33].

Theorem 1 (*Leibniz formula for Riemann–Liouville integrals [3, p. 75]*). *Let $\beta \in \mathbb{R}_+ / \mathbb{N}$, and assume that f is continuous on $[0, h]$ with some $h > 0$, and g is analytic on $[0, h]$. Then, the following formula holds*: $\forall t \in]0, h]$,

$$\mathrm{J}_t^{\beta} [f(t) g(t)] = \sum_{i=0}^{\infty} \binom{-\beta}{i} g^{(i)}(t) \mathrm{J}_t^{\beta+i} f(t), \tag{6}$$