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Perturbation analysis of simultaneous orthogonal matching pursuit

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ABSTRACT

The theory of compressed sensing (CS) indicates that a sparse vector lying in a high dimensional space can be accurately recovered from only a small set of linear measurements, under appropriate conditions on the measurement matrix. For multiple sparse signals that share common locations of the nonzero entries, simultaneous orthogonal matching pursuit (SOMP) is a widely used algorithm for joint recovery. In this paper, when both the measurements and the measurement matrix are perturbed by some errors, we analyze the performance of SOMP based on restricted isometry property (RIP). For an almost sparse signal ensemble $\{\mathbf{x}_j \in \mathbb{R}^N\}$, where the locations of the K ($K \ll N$) largest magnitude entries in each $\mathbf{x}_j$ are identical and the differences among each signal are not very large, the results reveal that the set of these locations can be recovered exactly under some RIP-based conditions. We prove that the derived conditions are rather tight for a special scenario. Furthermore, we extend the analysis to strong-decaying signal ensemble, where the decay of entries in each signal is sufficiently strong. The results show that the corresponding RIP-based conditions are relaxed when compared with arbitrary sparse signal ensemble.

1. Introduction

Recent theory of compressed sensing (CS) states that a K-sparse signal $\mathbf{x} \in \mathbb{R}^N$ can be represented by its fewer measurements in the form of

$$\mathbf{y} = \mathbf{\Phi x}, \tag{1}$$

where $\mathbf{y} \in \mathbb{R}^M$ is the measurement vector, and $\mathbf{\Phi} \in \mathbb{R}^{M \times N}$ is the measurement matrix ($K < M < N$) [1–3]. One fundamental problem in CS is to recover $\mathbf{x}$ from $\mathbf{y}$ based on the prior information of signal sparsity.

The recovery algorithms have received significant attention since the introduction of CS. Ref. [4] showed that the sparse signal can be stably recovered under some conditions on $\mathbf{\Phi}$ by solving an l_1-minimization problem. Other approaches, including subspace pursuit [5], orthogonal matching pursuit (OMP) [6], compressive sampling matching pursuit [7], and iterative hard thresholding [8], are also reported thereafter. Among them, OMP is a popular greedy algorithm, which has been studied widely. For the noiseless case in (1), [9,10] presented the conditions of exact recovery of $\mathbf{x}$ when OMP is used. When some additive noise exists in the system, which is assumed to be unknown and has a bounded norm, [11,12] analyzed the recovery accuracy of OMP. These existing theoretical results are mostly developed by using the tools of coherence [13] and restricted isometry property (RIP) [14].

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A variety of application scenarios motivate the researches to recover not just a single signal, but many correlated signals at the same time, which is called the joint recovery problem. It arises in many applications including magnetoencephalography [15], source localization [16], array processing [17], cognitive radio communications [18]. One typical scenario is to identify a common support shared by a set of signals from their measurement vectors, where the support is the set containing the locations of all nonzero entries in the signals [17]. When the measurement matrices are common for arbitrary signals, this problem is known as multiple measurement vectors (MMV).

There have been many studies on the MMV problem, see the discussions in [19–31] and the references in these papers. In terms of recovery algorithms, the methods include convex optimization [20–22], greedy pursuit [23–27], and sparse Bayesian learning [28,29]. In terms of theoretical guarantees, [20,22] analyzed the worst-case performance of MMV problem. However, the results did not show any performance gain with joint recovery. It is due to the reason that these results apply to all possible input signals, and the case of $\mathbf{x}_j = \mathbf{x}$ corresponds to no gain. In contrast, [24,30,31] provided the average-case analysis, which showed that fewer measurements are required when compared with separate recovery.

Among the literature on MMV, simultaneous OMP (SOMP), which can be regarded as an extension of OMP from one signal to multiple signals, is often used to perform joint recovery. In the context of MMV, SOMP is investigated when the measurement matrices for each signal ($\{\mathbf{\Phi}_j\}$) are identical [24,25]. However, the case of different $\mathbf{\Phi}_j$ is common in practical application. For example, to reduce the cost at the nodes implementing CS operation, it is advisable to generate $\mathbf{\Phi}_j$ independently without communication among them. Currently, the joint recovery problem when $\{\mathbf{\Phi}_j\}$ are different is seldom studied. The authors in [26,27] discussed this case, and proposed the corresponding reconstruction algorithms.

The measurement matrices are generally assumed to be known a priori in standard CS, see the analysis in [9–14, 24,25]. However, it is not always the case in practical situations. When various errors and fluctuations exist in the system, the general perturbations should be considered, which involve a perturbed measurement vector and a perturbed measurement matrix. The former, i.e., the measurement noise, has been discussed widely in many existing papers. For the latter, few works have been done. Refs. [32–36] considered the case of a single signal. Ref. [32] discussed the effect of a general matrix perturbation and showed that the recovery error of l_1-algorithm grows linearly with the perturbation level. Refs. [33] and [34] presented the degradation of signal recovery when perturbation exists due to the basis mismatch. Ref. [35] considered a structured matrix perturbation and showed that the recovery error of l_1-minimization is at most proportional to the measurement noise level. Ref. [36] derived the RIP-based recovery conditions of OMP when matrix perturbation is presented at either the encoder or decoder. Ref. [37] extended the case in [36] to the case of multiple sparse signals that share a common support, where it analyzed the OMP algorithm for MMV under general perturbations, and showed that the recovery of the support is guaranteed under certain conditions.

This paper is on the perturbed CS problem when SOMP is used to perform joint recovery for the case of different measurement matrices. The model to represent the correlation among signals is the one where the locations of the K largest magnitude entries in each signal are identical, and the set of corresponding position indices is defined as the support $\mathbf{\Omega}$. For an almost sparse signal ensemble, where the differences among each signal are not very large, we derive the RIP-based conditions to guarantee that $\mathbf{\Omega}$ can be exactly recovered. The error bound between each signal and its reconstruction is also derived. Furthermore, we extend the analysis to the signals that are strong-decaying, where the decay of entries in each signal is sufficiently strong, and derive the corresponding conditions for support recovery. When compared with the results in [37], our results enjoy three advantages. First, the analysis in this paper applies for the case of different measurement matrices. Second, we consider the almost sparse signals, which generalizes the case of exact sparse signals in [37]. Third, as shown in the experiments in Section 4, our results provide an improved bound when compared with the one in [37].

The rest of the paper is organized as follows. Section 2 gives the backgrounds of CS and perturbed CS. Section 3 develops the RIP-based conditions of SOMP in the perturbed scenario. Some experiments are given in Section 4. Section 5 discusses the derived results and concludes this paper. Finally, some mathematical proofs are provided in Appendices.

Notation: $\|\mathbf{A}\|_2$, $\|\mathbf{A}\|_F$ and $\|\mathbf{A}\|_2^{(K)}$ denote the spectral norm, Frobenius norm and the largest spectral norm taken over all K-column submatrices of the matrix $\mathbf{A}$, respectively. For a vector $\mathbf{v}$, $\mathbf{v}(i)$ means its ith entry. $\|\mathbf{v}\|_1 = \sum_{i=1}^{N} |\mathbf{v}(i)|$, and $\langle \mathbf{v}, \mathbf{u} \rangle = \sum_{i=1}^{N} \mathbf{v}(i)\mathbf{u}(i)$, where $\mathbf{v} \in \mathbb{R}^N$ and $\mathbf{u} \in \mathbb{R}^N$. supp($\mathbf{v}$) stands for the support of a sparse signal $\mathbf{v}$. $\mathbf{v}_{\mathbf{\Lambda}}$ denotes the vector obtained by selecting the entries of $\mathbf{v}$ indexed by $\mathbf{\Lambda}$, and $\mathbf{A}_{\mathbf{\Lambda}}$ is the matrix obtained by selecting the columns of $\mathbf{A}$ indexed by $\mathbf{\Lambda}$. For a matrix ensemble $\{\mathbf{A}_j\}$, $\mathbf{A}_{j,\mathbf{\Lambda}}$ is the matrix containing the columns of $\mathbf{A}_j$ indexed by $\mathbf{\Lambda}$, and $\mathbf{v}_{j,\mathbf{\Lambda}}$ for a vector ensemble $\{\mathbf{v}_j\}$ is defined likewise.

2. Backgrounds

2.1. Compressed sensing

Compressed sensing deals with the acquisition and recovery of sparse signals from a small number of random linear projections. It is well known that $\mathbf{x}$ can be reconstructed by solving the l_1-minimization problem [14]

$$\min_{\mathbf{x}} \|\mathbf{x}\|_1 \text{ s.t. } \mathbf{\Phi x} = \mathbf{y}.$$

To study the performance of CS recovery algorithms, RIP is used as an important tool [14]. We say that a matrix $\mathbf{\Phi}$ satisfies the RIP of order K if

$$(1-\delta_K)\|\mathbf{x}\|_2^2 \le \|\Phi \mathbf{x}\|_2^2 \le (1+\delta_K)\|\mathbf{x}\|_2^2$$

for every K-sparse signal $\mathbf{x}$ with the restricted isometry constant (RIC) being $0 < \delta_K < 1$. It has been shown that $\mathbf{x}$ can be perfectly recovered by solving l_1-minimization problem if $\delta_{2K} < \sqrt{2} - 1$ [14], which is further improved to $\delta_{2K} < 0.4931$ in [38].