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Integer/fractional decomposition of the impulse response of fractional linear systems

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ABSTRACT

The decomposition of a fractional linear system is discussed in this paper. It is shown that it can be decomposed into an integer order part, corresponding to possible existing poles, and a fractional part. The first and second parts are responsible for the short and long memory behaviors of the system, respectively, known as characteristic of fractional systems.

1. Introduction

Fractional linear systems are usually expressed by a fractional order constant coefficient linear differential equation with the general format [1,2]:

$$\sum_{k=0}^{N} a_k D^{\alpha_k} y(t) = \sum_{k=0}^{M} b_k D^{\beta_k} x(t) \tag{1}$$

where the symbol D represents the derivative operator, $t \in \mathbf{R}$, or $t \in \mathbf{N}$, if the system is continuous-time, or discrete-time differential. The parameters α_k and β_k denote the derivative orders that we assume to form strictly increasing sequences of positive numbers, and $a_k, b_k \in \mathbf{R}$. In the so-called commensurate case we write $\alpha_k = \beta_k = k\alpha,\ k \in \mathbf{N}$, with $0 < \alpha \le 1$. In current applications we assume that $\beta_M \le \alpha_N$ for stability reasons.

There are several definitions of derivative [3,4]. Here we will assume that we are dealing with causal derivatives such that

$$D_f^{\alpha} e^{st} = s^{\alpha} e^{st} \quad \text{if } \mathrm{Re}(s) > 0 \tag{2}$$

In which concerns causal derivative definitions the most suitable is the forward Grünwald–Letnikov derivative:

$$D_f^{\alpha} f(t) = \lim_{h \to 0^+} \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(t-kh). \tag{3}$$

Relations (2) and (3) are valid for any order, but we will assume that α is positive, if the limit in (3) exists for the considered function. According to (2) the exponential

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is the eigenfunction of the system defined by (1). The corresponding eigenvalue, $H(s)$, is the transfer function (TF) of [2]

$$H(s)=\frac{B(s)}{A(s)}=\frac{\sum_{k=0}^{M} b_k s^{\beta_k}}{\sum_{k=0}^{N} a_k s^{\alpha_k}} \tag{4}$$

As in the classical case we will name by "poles" the roots of the *characteristic pseudo-polynomial*, $A(s)$, in the denominator of the TF [1].

The main objective of this paper is the inversion of (4), in order to get the impulse response. The inversion of each partial fraction is currently done by means of the Mittag-Leffler function [3,1]. We proposed an alternative method, by decomposing the impulse response into two terms that are obtained using the Hankel integration path [5,1] to compute the Bromwich integral. One term corresponds to the integer order part and results from the residue theorem. The other term is the purely fractional part that assumes an integral form. This is a generalization of a more restrict result obtained by Gorenflo and Mainardi [6,7]. With this result we can show that the impulse response, corresponding to (10), has two contributions.

The general formulation (10) is not easy to tackle. With the help of the Laplace transform (LT) it is possible to obtain the impulse response using a rather involved expression [3]. Nevertheless, these approaches have an important drawback: the solutions rely on one, or several, Taylor or Mittag-Leffler series, that create several computational problems. Furthermore, these approaches mask the underlying structure of the system, in the sense that they do not highlight the presence of two different terms:

- One component of integer order that inherits the classical behavior, mainly oscillations and (un)stability.
- One component of fractional order responsible for the long range behavior of the fractional linear systems, that is intrinsically stable as we will demonstrate in the sequel.

Having these ideas in mind, the paper is organized as follows. In Section 2 we describe the referred decomposition and we show how to compute each part. In Section 3 we illustrate the decomposition procedure. Finally, in Section 4 we outline the main conclusions.

2. The inversion of the transfer function

The solution supplied by Taylor, Mittag-Leffler, or some series of the same type masks the underlying structure of the TF. This limitation is revealed when we try to compute its inversion by using the Bromwich integral, or the Mellin's inverse formula. In fact, to obtain it we must fix a branch cut line. As the transform must be analytic on the right half complex plane we choose the left half real axis. On the other hand, function (4) is continuous from above on the branch cut line and verifies $\lim_{s\to\infty} H(s)=0$, $|\arg(s)|<\pi$. We will assume that $\lim_{s\to 0} sH(s)=0$. Let (γ_k, p_k), $k=1,2,\ldots,K$, be the pairs (order, root) such that $A(s)=\prod_1^K (s^{\gamma_k}-p_k)$. Let $K_0 \le K$ be the number of poles. We remember that a given root p, corresponding to a given order γ, is a pole, if when $s=|s|e^{i\theta}$ and $p=|p|e^{i\phi}$, we have $|s|=|p|^{1/\gamma}$ and $\theta=\phi/\gamma$. However, we have $|\phi|\le\pi$ and, therefore, we only obtain a pole if $|\phi|\le\pi/\gamma$.

In these conditions we can use the Hankel integration path [5,1] and we apply the residue theorem. Let $u\in\mathbf{R}^+$ and consider $H(e^{i\pi}u)$ and $H(e^{-i\pi}u)$, the values of $H(s)$ immediately above and below the branch cut line. Proceeding as in [5] we obtain

$$h(t)=\sum_{k=1}^{K_0} A_k e^{p_k^{1/\gamma_k} t}\varepsilon(t)+\frac{1}{2\pi i}\int_0^\infty \left[H(e^{-i\pi}u) - H(e^{i\pi}u)\right] e^{-\sigma t}\,du\cdot\varepsilon(t) \tag{5}$$

where $\varepsilon(t)$ is the unit step function and the constants A_k, $k=1,2,\ldots,K_0$, are the residues of (4) at p_k^{1/γ_k}.

This expression generalizes a more restrict result obtained by Gorenflo and Mainardi, [6–8], in deducing the properties of Mittag-Leffler function.

Computing the LT of both sides in (5) we obtain

$$H(s)=H_i(s)+H_f(s) \tag{6}$$

where the integer order part is

$$H_i(s)=\sum_{k=1}^{K_0}\frac{A_k}{s-p_k^{1/\gamma_k}},\quad \mathrm{Re}(s)>\max\left(\mathrm{Re}\left(p_k^{1/\gamma_k}\right)\right) \tag{7}$$

and the fractional part is

$$H_f(s)=\frac{1}{2\pi i}\int_0^\infty \left[H(e^{-i\pi}u)-H(e^{i\pi}u)\right]\frac{1}{s+u}\,du \tag{8}$$

valid for $\mathrm{Re}(s)>0$.

Fig. 1 shows the amplitude (in logarithmic scale) and phase (in linear scale) spectra for $0\le\alpha\le 1$ and $p=-1$.

The above steps led us to realize that

- For $\gamma_k=1$, $k=1,2,\ldots,K$, we have no fractional component. The TF is a sum of partial fractions and each one has an exponential for solution.
- For $\gamma_k<1$, $k=1,2,\ldots,K$, we may have two components depending on the location of p_k in the complex plane
  - If $|\arg(p_k)|>\pi/\gamma_k$, $k=1,2,\ldots,K$, then we do not have the integer order component; it is a purely fractional system.
  - If $|\arg(p_k)|\le\pi/\gamma_k$, $k=1,2,\ldots,K$, for some k, then it is mixed character system in the sense that we have both components.
  - When $|\arg(p_k)|=\pi/(2\gamma_k)$, for some k, the integer order component is sinusoidal; however, the fractional component decreases to zero.
- The stability condition comes only from the integer order component. In fact, and as it is straightforward to verify, the integer order component is stable if $\pi/(2\gamma_k) < |\arg(p_k)|<\pi/\gamma_k$, $k=1,2,\ldots,K_0$, and unstable if $|\arg(p_k)|<\pi/(2\gamma_k)$, $k=1,2,\ldots,K_0$. The case $|\arg(p_k)|=\pi/(2\gamma_k)$ corresponds to a critically stable system.

Concerning to the fractional part we can verify that $H(e^{-i\pi}\sigma)-H(e^{i\pi}\sigma)$ is a bounded function. Therefore, the integral in (5) is also bounded and decreases to zero as t goes to infinite.

In the sequel we will assume that we are dealing with stable systems. The above considerations allow us to