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Least squares generalized B-spline signal and image processing

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Abstract

This paper presents generalized B-spline interpolation and approximation techniques. The first part of this paper describes a technique for the continuous representation of discrete signals in terms of generalized B-splines (direct B-spline transform) and for interpolative signal reconstruction (indirect B-spline transform) with an expansion factor *m*. The key to our approach is to use a family of generalized B-spline filters. It has been shown that the scale factor can be modified to tune the approximated filter bandwidths. The second part of this paper makes use of these results to describe an equivalent filtering interpretation (low-pass filtering followed by a generalized B-spline interpolator) of generalized B-splines together with least squares generalized B-spline signal approximation methods. This part also concern with new theoretical results showing factorization of the transfer functions of the higher degree ($n > 3$) generalized B-spline filters. Experimentally, it is seen that data compression with reduced amount of error can be achieved using these least squares generalized B-spline filtering techniques.

Keywords: Generalized B-spline filtering; Least squares generalized B-splines; Signal processing; Image processing; Recursive filtering

1. Introduction

B-spline signal representation and interpolation methods are found increasingly popular due to the advent of multiresolution techniques [4,5]. Unser et al. have described the general case of B-spline interpolation of any order and have provided mechanisms for the efficient design of filters to evaluate the direct or indirect B-spline transforms [1,8–11]. Unser et al. have also presented the theory of least squares cubic B-spline signal approximation technique which is an extension of the standard least squares approximation technique proposed by Carl de Boor [3], and they have used this technique for different image processing applications [9,10]. We have been motivated to develop a framework for solving the classical problems of generalized B-spline interpolation and approximation using digital filtering techniques. Recently, we have proposed a generalized filtering technique for B-spline coefficient extraction and interpolative signal reconstruction [6]. The main purpose of this paper is to extend these recent results to image processing applications based on the

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representation of a signal in terms of continuous generalized B-spline basis functions.

The whole family of such generalized B-spline functions for degree $n = 0,1,2,3$ are derived and presented in this paper. In the first half of this paper, we propose a scheme for decimation and interpolation of discrete sequences in a signal processing point of view. The entire family of direct and indirect generalized B-spline filters of degree $n = 0,1,2,3$ are derived and presented in this paper. New theoretical results connected with the factorization of higher degree ($n > 3$) generalized B-spline filters are presented.

In the second part, we present the least squares signal approximation technique, proposed in the brilliant work by Unser et al. [9,10], replacing the least squares cubic spline operators by least squares generalized cubic spline operators. The whole family of least squares generalized B-spline filters for degree $n = 0,1,2,3$ are derived and presented in this paper. The advantage of such a scheme is to tune the filter bandwidth by changing the filter parameter α (scale factor). The quality of interpolation is interpreted from the frequency domain analysis. This paper presents some experimental results on image data compression. These experimental results show that the use of least squares generalized B-spline operators yields improved fidelity and SNR.

The organization of the paper is as follows. In Section 2, we review some essential properties of continuous B-spline functions, which also includes some extensions of our previous results [6]. Section 3 contains signal and image processing techniques using generalized B-spline filters. In Section 4, we describe least squares signal approximation techniques using least squares generalized B-spline filters.

2. Preliminaries

2.1. Continuous L_2 polynomial splines and discrete B-spline functions

The piecewise polynomial spline functions are used to approximate or interpolate a given sequence $a(k) \in l_2$ (where l_2 is the Hilbert space of square-summable real-valued sequences), maintaining continuity of derivatives at the knots. Let $\mathscr{S}_m^n$ be the subspace of polynomial splines of degree n with equally spaced knots and m denotes the spacing between knots. Hence, $\mathscr{S}_m^n \subset L_2$ (where L_2 is the Hilbert space of measurable square-integrable one-dimensional functions), and are of class $\mathscr{C}^{n-1}$. There also exists a basis function known as B-spline which can be efficiently used for interpolation [3,7]. The normalized B-spline functions of order n with $(n+2)$ equally spaced knots are given as [7]

$$\beta^n(x) = \frac{1}{n!}\sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} (x-j)^n U(x-j), \quad (2.1)$$

where $\binom{n+1}{j}$ are the binomial coefficients, and $U(x)$ is the unit step function. $\beta^n(x)$ is the symmetrical B-spline of order n. The convolution property of symmetrical B-spline functions can be written as

$$\beta^n(x) = \beta^p * \beta^q(x), \quad (2.2)$$

where $n = p + q + 1$ with $p, q \geqslant 0$. Unser et al. developed a digital signal processing framework for B-splines [9,10]. The same ideas will be applied here for signal representation and interpolation in terms of generalized B-splines.

2.2. Basic concept of generalized B-splines

In our earlier paper [6], we have introduced a new concept of generalized B-splines that provide compact support and have continuous $(n-1)$th order derivatives. A family of generalized B-splines were generated using the basic properties of B-spline functions as discussed in the preceding section. These generalized B-splines of order n support $n+2$ data points. A generalized B-spline of degree n and type r expanded by an integer factor m is given by

$$B_m^{nr}(x) = L\sum_{j=0}^{n+1} W_j g_m^{nr}(x - jm) U(x - jm), \quad (2.3)$$

where L is the constant of normalization, $g_m^{nr}(x)$ is the (piecewise interpolant) B-spline generating function of order n and type r expanded by an integer factor of m, $U(x)$ is the unit step function and W_j is the constant of multiplication in the jth