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An improved fractional-order differentiation model for image denoising

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This paper investigates fractional order differentiation and its applications in digital image processing. We propose an improved model based on the Grünwald-Letnikov (G–L) fractional differential operator. Our improved denoising operator mask is based on G–L fractional order differentiation. The total coefficient of this mask is not equal to zero, which means that its response value is not zero in flat areas of the image. This nonlinear filter mask enhances and preserves detailed features while effectively denoising the image. Our experiments on texture-rich digital images demonstrated the capabilities of the filter. We used the information entropy and average gradient to quantitatively compare our method to existing techniques. Additionally, we have successfully used it to denoise three-dimensional magnetic resonance images.

1. Introduction

Images are often corrupted by additive noise when they are being captured and transmitted. The main aim of an image denoising algorithm is to reduce the noise level while preserving the image features. In recent years, fractional calculus has become increasingly important to foundational research and engineering applications. Fractional calculus provides methods to differentiate and integrate functions to non-integer orders. The concept of fractional calculus came into existence in 1695 during discussions between Leibniz and L'Hospital. Three popular definitions for fractional calculus were given by Grunwald–Letnikov (G–L), Riemann–Liouville (R–L), and Caputo [1–3]. Of these, G–L and R–L are the most popular definitions used in digital image processing. The G–L-based differential operator is used by many researchers and scholars [4–6]. Fractional differential is an effective mathematical method for dealing with fractal problems [7–9]. Fractional differentiation for image processing is a burgeoning subject [10–12]. Pu et al. proved that fractional differential-based methods can preserve low-frequency contour features in smooth areas. They also proved that they retain high-frequency marginal features in areas that have large gray-level variabilities, and can enhance texture details in areas that do not have significant gray-level variabilities [13–16].

Three-dimensional (3-D) images are becoming increasingly common in image processing applications [17–21]. For instance, magnetic resonance images (MRIs) and functional MRIs (fMRIs) are used to study the biological mechanism of a 3-D object (e.g., a patient's ankle). They acquire a set of two-dimensional (2-D) images that correspond to slices of the 3-D object. Then, the 3-D object is reconstructed from the 2-D images. This is called 3-D image reconstruction. However, 3-D images often contain

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noise because of hardware imperfections and other reasons. 3-D image denoising based on minimizing the total variation (TV) [22] is popular in image processing. Chambolle's algorithm has recently been developed to denoise 3-D images [23]. Other 3-D image denoising methods are based on 3-D sparse representations [24,25], non-local means [26], and other techniques.

In this paper, we propose a fractional order image denoising method that can preserve edges and major edge features. Our method consists of three major steps. First, it uses the G–L definition as the basis of our fractional order differential equation-based denoising method. Second, we derive an improved G–L image denoising mask and numerical method. Finally, we demonstrate the denoising capability of the proposed method. Our experimental results prove that the algorithm can preserve low-frequency contour features in a smooth area. Additionally, it is a nonlinear method that maintains high-frequency edge and texture details in areas where the gray level does not significantly vary.

The outline of this paper is as follows. In Section 2, we introduce work related to the G–L fractional order derivative definition and properties. In Section 3, we develop an improved G–L method to solve the image denoising problem. Numerical examples are presented in Section 4, and the paper is concluded in Section 5.

2. Related work

The G–L, R–L, and Caputo definitions are the most commonly used definitions of fractional calculus for the Euclidean measure. The R–L and Caputo definitions use the Cauchy equation, so they are computationally complex. The G–L definition expresses a function using a weighted sum around the function. It is appropriate for image processing applications. According to [16], the G–L defined ν-order differential of signal $s(t)$ is

$$ {}_{a}^{G}D_{t}^{\nu s(x)} \triangleq \lim_{h\to 0} h^{-\nu}(-1)^m \sum_{m=0}^{n-1} \frac{\Gamma(\nu+1)}{\Gamma(m+1)\Gamma(\nu-m+1)} s(t-mh), \quad (1) $$

where the duration of $s(t)$ is $[a,t]$, $\nu \in R$ (and may be a fraction), $h=(t-a)/(n)$ is the step size, and $\Gamma(x)=(x-1)!$ is the gamma function of x. Eq. (1) shows that the G–L definition in the Euclidean measure extends the step from integer to fractional numbers, and thus it extends the concept from integer to fractional differentiation. G–L defined fractional calculus can be easily calculated. It only depends on the discrete sampling value of $s(t-k((t-a)/N))$ (which correlates to $s(t)$) and is not related to the value of the derivative or integral.

Next, we analysis the influence that fractional order differentiation has on a signal. The Fourier transform of the signal $s(t)$ is

$$ \mathrm{FT}\left[D^{\nu}s(x)\right] = (i\omega)^{\nu}\mathrm{FT}[s(t)] - \sum_{k=0}^{n-1}(i\omega)^{k}\frac{d^{\nu-1-k}}{dx^{\nu-1-k}}s(0), \quad (2) $$

where i denotes the imaginary unit, and ω represents the digital frequency.

The amplitude characteristic is an even function and the phase characteristic is an odd function. We have analyzed the characteristics of the fractional differential filter for $\omega > 0$. The frequency response of the fractional differential filter is given in Fig. 1.

We have observed that the frequency response to fractional differential filter is nonlinear when $v=0$. Additionally, the v-order fractional differential is an all-pass filter and its frequency response is $\hat{d}^{v}(\omega)=(i\omega)^{v}$. When $v<0$, the filter is a fractional integrator and a singular low-pass integral filter, as shown in Fig. 1(b). When $v>0$, it is a fractional derivative operator and its frequency response is $\lim_{|\omega|\to\infty}\left|\hat{d}^{v}(\omega)\right|\to\infty$. Here, $\hat{d}^{v}(\omega)$ is a singular high-pass differential filter (see Fig. 1(a)). If v increases, the transmission bands of $\hat{d}^{v}(\omega)$ become narrower and the high-pass characteristic is stronger. In the low-frequency section of $0<\omega<1$, the preservation magnitude of low-frequency contours using fractional differential is superior to that by a first-order derivative. We consider that the gray scale does not significantly change in an image's smooth area. The texture features in a smooth area may be significantly attenuated and its differential result may be nearly zero. For this reason, the integral differential linearly attenuates the texture features, and cannot preserve them in these

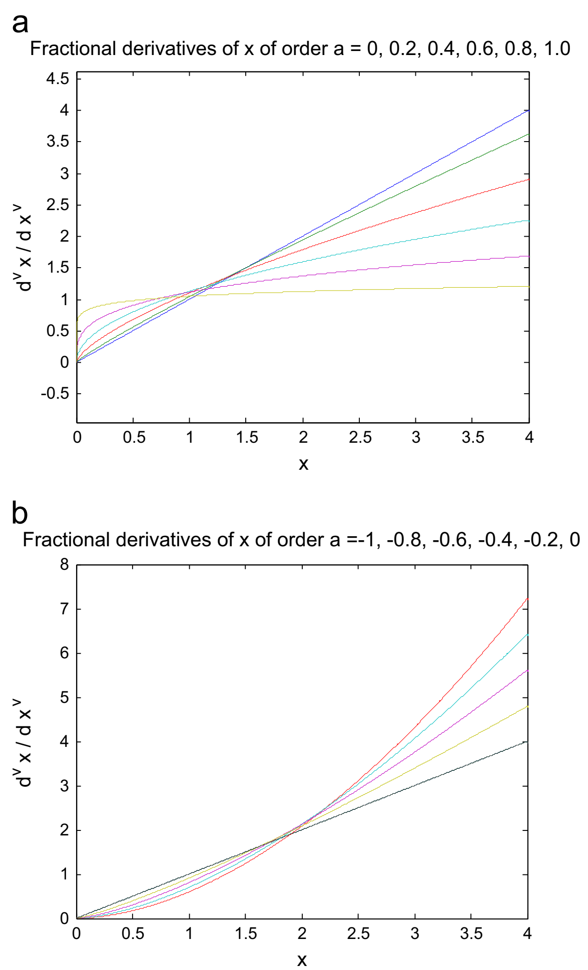

Fig. 1. Frequency response of the fractional differential filter: (a) positive and (b) negative orders.