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A heuristic search algorithm for the multiple measurement vectors problem

Xinpeng Du [a,*], Lizhi Cheng [b,c], Guangquan Cheng [d]

[a] State Key Laboratory of astronautic Dynamics, Xi'an Satellite Control Center, Xi'an, 710043 Shanxi, China
[b] Department of Mathematics and System Science, College of Science, National University of Defense Technology, Changsha, 410073 Hunan, China
[c] The State Key Laboratory for High Performance Computation, National University of Defense Technology, Changsha, 410073 Hunan, China
[d] Science and Technology on Information Systems Engineering Laboratory, College of Information System and Management, National University of Defense Technology, Changsha, 410073 Hunan, China

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ABSTRACT

In this paper, we address the multiple measurement vectors problem, which is now a hot topic in the compressed sensing theory and its various applications. We propose a novel heuristic search algorithm called HSAMMV to solve the problem, which is modeled as a combinatorial optimization. HSAMMV is proposed in the framework of simulated annealing algorithm. The main innovation is to take advantage of some greedy pursuit algorithms for designing the initial solution and the generating mechanism of HSAMMV. Compared with some state-of-the-art algorithms, the numerical simulation results illustrate that HSAMMV has strong global search ability and quite good recovery performance.

1. Introduction

The sparse recovery problem is of great importance in compressed sensing (CS) theory [1–3], and it focuses on finding the sparsest possible solution of the underdetermined linear system of equations $Ax=b$ for given matrix $A \in \mathbb{R}^{m\times n}$ $(m<n)$ and vector $b \in \mathbb{R}^m$. The problem can be written as

$$\min_x \|x\|_0 \quad \text{s.t.} \quad Ax=b \tag{1}$$

where $\|x\|_0$ denotes the number of nonzero entries of x.

There is only a single measurement (i.e., the vector b) in problem (1), so it is referred to as the single measurement vector (SMV) problem. A natural extension of SMV, multiple measurement vectors (MMV, also called joint sparse recovery), attracts increasing attention of the research community and can be used in source localization [4–6], multi-task learning [7] and neuro-magnetic imaging [8] etc. In MMV, we are given multiple measurements $B \in \mathbb{R}^{m\times l}$ with the number of snapshots $l>1$, and aim to solve the linear system of equations $AX=B$ in which X is supposed to be jointly sparse (i.e., only a few rows are nonzero). The noiseless MMV can be modeled as

$$\min_X |\Re(X)| \quad \text{s.t.} \quad AX=B \tag{2}$$

where $\Re(X) \triangleq \{1 \le i \le n | X_{i,\cdot} \ne \mathbf{0}\}$ denotes the row support of X, $X_{i,\cdot}$ denotes the i-th row of X, and $|\cdot|$ represents the cardinality of a set.

In [8–10], the authors have proved that a solution X of $AX=B$ is the unique solution of (2) if

$$|\Re(X)| < \frac{\text{spark}(A)+\text{rank}(B)-1}{2} \tag{3}$$

* Corresponding author.
E-mail addresses: duxp1984@gmail.com (X. Du), clzcheng@nudt.edu.cn (L. Cheng), cgq299@163.com (G. Cheng).

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where the quantity spark(A) denotes the smallest number of columns from A that are linearly dependent.

The model (2) is noiseless, and it is oversimple. In practice, it generally cannot avoid modeling error and measurement error. And we often modify the model with additive noise to deal with both situations. An MMV with additive noise can be stated as

$$\min_X |\Re(X)| \quad \text{s.t.} \quad \|AX-B\|_F \le \varepsilon \tag{4}$$

where $B = AX + N$, N is the additive noise and $\varepsilon \ge 0$ is the error bound. It is obvious that problem (2) is a particular case of problem (4) with $N = \mathbf{0}$.

Problem (4) is NP-hard in general [2], and many efficient algorithms have been proposed to solve it, such as l_1-SVD [4], M-FOCUSS (FOCal Underdetermined System Solver for MMV) [8], M-OMP (Orthogonal Matching Pursuit for MMV) [9], ReMBo (Reduce MMV and Boost) [10], RA-ORMP (Rank Aware Order Recursive Matching Pursuit) [11], CS-MUSIC (Compressive MUSIC) [12], RPMB (Randomly Project MMV and Boost) [13], q-thresholding algorithm ($q \ge 1$) [14], SA-MUSIC (Subspace-Augmented MUSIC) [15], T-MSBL (a Temporal extension of the Sparse Bayesian Learning algorithm for MMV) [16], AMPMMV (Approximate Message Passing based MMV algorithm) [17], and ZAPMMV (Zero-point Attracting Projection algorithm for MMV) [18].

Although the existing algorithms can achieve satisfactory recovery under specific conditions, they perform well only when the number of snapshots is relatively large or the sparsity level is relatively small [12,13]. Moreover, many of the existing algorithms perform unsatisfactorily in the rank-defective case (i.e., rank$(X) < |\Re(X)|$ [15]), such as M-OMP, M-FOCUSS and l_1-SVD.

One main reason resulting in the aforementioned shortcomings is that the existing MMV algorithms often produce sub-optimums of problem (4). The sparse recovery problem is essentially a combinatorial optimization [2], and we know that the simulated annealing (SA) algorithm [19] is very efficient in finding the global optimums for the combinatorial optimization problems. Therefore, we take advantage of SA to propose a new MMV algorithm to overcome the shortcomings in this paper. We first model the MMV problem as a combinatorial optimization, and then propose a novel heuristic search algorithm termed HSAMMV to solve the modeled problem based on SA and some existing CS algorithms. In HSAMMV, the initial solution is designed using the q-thresholding algorithm ($q \ge 1$) [14], and the generating mechanism is designed using the pruning technique existed in SP (Subspace Pursuit) [20] and CoSaMP (Compressive Sampling Matching Pursuit) [21]. Compared with some state-of-the-art algorithms, the numerical simulation results illustrate that HSAMMV has strong global search ability and quite good recovery performance. Specifically, HSAMMV still performs well when the number of snapshots is relatively small or the sparsity level is relatively large, and it is effective in the rank-defective case and has a robust performance to the sparsity level. In a word, HSAMMV can well overcome the aforementioned shortcomings of the existing MMV algorithms to some extent.

Throughout the paper, we use the following notations. For any matrix $M \in \mathbb{R}^{m\times n}$, $M_{i,\cdot}$ denotes the i-th row of M, and $M_{\cdot,j}$ denotes the j-th column of M. The row support of M is defined as $\Re(M) \triangleq \{1 \le i \le n | M_{i,\cdot} \ne \mathbf{0}\}$. For a column full-rank matrix H, its pseudo-inverse is defined by $H^\dagger = (H^T H)^{-1} H^T$, where '$T$' represents matrix transposition. $|\cdot|$ represents the cardinality of a set, and $\lfloor \cdot \rfloor$ denotes the flooring operation for a real number (i.e., $\lfloor \varsigma \rfloor$ equals to the nearest integer less than or equal to ς). The matrix M is called K-jointly sparse if $|\Re(M)| \le K < m$, where K is called the sparsity level of M. $\|M\|_F$ denotes the Frobenius norm of the matrix M, and $\|x\|_q$ ($q \ge 1$) represents the l_q norm of the vector $x \in \mathbb{R}^n$. Suppose $G \subseteq \{1, 2, \ldots, n\}$ is a nonempty subset, the vector x_G consists of the entries indexed by $i \in G$, the matrix M_G is composed of the columns $\{M_{\cdot,j}\}_{j\in G}$, and the matrix $M_{G,:}$ denotes a matrix composed of the rows $\{M_{j,\cdot}\}_{j\in G}$.

The rest of the paper is organized as follows. In Section 2, we propose the HSAMMV algorithm and give some theoretical analysis. Simulation results are reported in Section 3, and the conclusions are drawn in Section 4.

2. HSAMMV: a heuristic search algorithm for MMV

In this section, we first give a brief introduction to SA in Section 2.1, next design the main elements of HSAMMV in Section 2.2, later give the computational complexity analysis of HSAMMV in Section 2.3, and then compare HSAMMV with some existing works in Section 2.4.

2.1. A short review of SA

Simulated annealing (SA) algorithm was proposed by Kirkpatrick et al. [19] in 1983 based on the annealing of metals. The main advantage of SA is the ability to avoid being stuck at local minimums because it permits accepting a less optimal solution (as compared with the current one) with a positive probability, and hence it has strong global search ability [22–24]. SA has attracted much attention due to its success in solving several large scale and complex problems, including some NP-complete problems such as Traveling Salesman Problem (TSP) [25] and the Flow Shop Scheduling Problem (FSSP) [26].

Consider a minimization problem $\min_{t\in\Omega} g(t)$, where Ω is the solution set and $g: \Omega \to \mathbb{R}$ is a cost function, the procedure of SA is summarized in Algorithm 1. In the algorithm, the generating mechanism is a method to select a new solution from the neighborhood of the current solution; and the annealing schedule is a sequence of positive real numbers, which are decreasing to zero, for setting temperatures.

Algorithm 1. *SA* [27] for solving $\min_{t\in\Omega} g(t)$.

Initialization: the initial solution $t_0 \in \Omega$, the number of the outer-loop iteration $k=1$.

The k-th outer-loop iteration ($k \ge 1$):

Step 1: (The inner-loop at T_k) First, set $t_k^0 = t_{k-1}$ and $j=1$; next, repeat steps 1.1–1.3 until some inner-loop stopping criterion is met and obtain t_k; then goto step 2.

Step 1.1: Randomly generate a neighboring solution y_k^j of the current solution t_k^{j-1} according to some generating mechanism and compute $g(y_k^j)$.

Step 1.2: Calculate $\Delta_k^j = g(y_k^j) - g(t_k^{j-1})$, and accept y_k^j if $p_k^j \ge \eta$, where η is a random number uniformly chosen in [0, 1] and $p_k^j = \min(1, \exp(-\Delta_k^j / T_k))$.