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# Parameter estimation of superimposed damped sinusoids using exponential windows

M. Ali Al-Radhawi [a,*], Karim Abed-Meraim [b]

[a] Department of Electrical and Electronic Engineering, Imperial College London, London SW7 2AZ, United Kingdom
[b] PRISME Laboratory, Polytech'Orléans, Orléans University, 45072 Orléans, France

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ABSTRACT

This paper presents a preprocessing technique based on exponential windowing (EW) for parameter estimation of superimposed exponentially damped sinusoids. It is shown that the EW technique significantly improves the robustness to noise over two other commonly used preprocessing techniques: subspace decomposition and higher order statistics. An ad hoc but efficient approach for the EW parameter selection is provided and shown to provide close to CRB performance.



## 1. Introduction

Parameter estimation of superimposed exponentially damped sinusoids from a set of noisy observation data is a problem one frequently encounters in many applications such as nuclear magnetic resonance imaging [1], power systems [2], audio modeling [3], mechanical systems [4], physics [5], and chemistry [6]. This problem has received considerable attention in the signal processing community in the last decades and is now considered by many to be “solved”. Indeed, there are plenty of “solutions” available to this problem. The best known methods include Prony's method [7,8], the MUSIC method [9], the ESPRIT method [10], the iterative quadratic maximum likelihood (IQML) method [11,12], the matrix pencil (MP) method [8], and state space-based method [13]. In addition to the above methods, there are many tools available to preprocess (i.e., clean up) the raw data before any of the above methods is applied. The most popular ones of these tools are subspace decomposition (SD) [7,8] and higher order statistics (HOS) [14,15].

In the last few years, renewed interest in the sinusoidal estimation problem has been observed either for dedicated applications [1,3] or for adverse situations such as irregular sampling [16], impulsive noise [17], colored noise [18], multiple dimensions [19], and multiple channels [20].

In this work, we consider a colored noise context and we discuss the idea of preprocessing the data with complex exponential windowing. Although there are various ways to modify and improve Prony's method, or the MP method with a preprocessing step, using SD or HOS is known to be the most effective in reducing the noise effect. However, the SD method, which separates “signal subspace” from “noise subspace”, assumes that the noise is either white, or colored with known covariance up to a scalar. The HOS method works well only with Gaussian noise and long data length.

We show that the concept of cyclostationarity (CS) [21,22] or more generally exponential windowing (EW)

* Corresponding author.
*E-mail addresses:* m.rizvi11@imperial.ac.uk (M. Ali Al-Radhawi), karim.abed-meraim@univ-orleans.fr (K. Abed-Meraim).

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can be easily applied to better preprocess the superimposed exponentially damped sinusoids. The CS or EW methods preserve the signal structure but mitigate the noise effectively as long as it is stationary. Compared to the SD and HOS methods, the CS or EW methods require the least assumption on the noise and hence perform the best. This advantage will be clearly illustrated by simulation results. We propose simple approaches to optimize the EW parameters in such a way that the signal power or the post-processing signal to noise ratio (SNR) is maximized. As a benchmark, we derive the Cramér–Rao bound (CRB) expression and use it to illustrate the closeness of the proposed method′s performance to the CRB.

The organization of this paper is as follows. In Section 2, the idea behind using cyclo-stationarity for pre-processing and a generalized exponential windowing method are introduced. Section 3 is devoted to performance investigations. First, we review the CRB expression for the colored noise case. Then, we evaluate the asymptotic covariance matrix of the pre-processed noise term, which is evaluated for white noise and used to express the averaged signal-to-noise ratio (after data pre-processing) that is used for efficient EW parameter selection. In Section 4, simulation results are presented demonstrating the superiority of the proposed method. Finally, we state the conclusion.

## 2. Proposed signal preprocessing methodology

### 2.1. Data model

The data $y(n)$ under consideration is modeled as follows:

$$y(n) = x(n) + w(n) = \sum_{m=1}^{L} h_m e^{b_m n} + w(n), \quad n = 0, 1, \ldots, N-1 \tag{1}$$

where $h_m = a_m e^{j\theta_m}$, $b_m = \alpha_m + j\omega_m$ with $\alpha_m < 0$, and $w(n)$ denotes the stationary noise.[1] $a_m$ and $\theta_m$ are respectively the amplitude and the initial phase of the $m$th sinusoid; its damping and frequency factors are respectively $\alpha_m$ and $\omega_m$.

### 2.2. Preprocessing using cyclo-stationarity

Considering the fact that exponentially damped signals have relatively short (effective) length, we apply the CS concept only in the context of second order statistics (SOS) as opposed to HOS. Define the $k$th lag ($k \geq 0$) cyclo-correlation $R_\beta(k)$ at the cyclo-frequency $\beta$ as

$$R_\beta(k) := \frac{1}{N} \sum_{n=0}^{N-k-1} y(n+k) y^*(n) e^{j\beta n} \tag{2}$$

where * denotes the complex conjugate transpose operator. Assuming large sample size,[2] and using (1) in (2) yield

$$r_\beta(k) = \frac{1}{N} \sum_{m,l=1}^{L} h_m h_l^* e^{k b_m} \sum_{n=0}^{N-1-k} e^{(b_m + b_l^* + j\beta)n} + \mathbb{E}v(k)$$

$$\approx \frac{1}{N} \sum_{m=1}^{L} A_\beta(m) e^{k b_m} + \mathbb{E}v(k) \tag{3}$$

where $r_\beta(k) = \mathbb{E}R_\beta(k)$, and

$$v(k) = \frac{1}{N} \sum_{n=0}^{N-1-k} w(n+k) w^*(n) e^{j\beta n},$$

$$A_\beta(m) = \sum_{l=1}^{L} \frac{h_m h_l^*}{1 - e^{b_m + b_l^* + j\beta}}, \quad m = 1, \ldots, L. \tag{4}$$

It is seen from (4) that the cyclo-correlation as a function of $k$ also consists of superimposed exponentially damped sinusoids and a noise term. The main advantage, in dealing with cyclo-correlation instead of correlation function as considered in standard sinusoidal estimation techniques [8,11], is that noise contribution is considerably reduced in the former case. In fact, due to the stationarity assumption of the noise process, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1-k} w(n+k) w^*(n) e^{j\beta n} = 0$$

when

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1-k} w(n+k) w^*(n) = \rho(k)$$

$\rho(k)$ being the $k$th correlation factor of the noise process. Generally, for additive colored noise, the signal to noise ratio (SNR) gain can be considerable since, for $\rho(k) \neq 0$, we have

$$\lim_{N\to\infty} \left| \frac{\sum_{n=0}^{N-1-k} w(n+k) w^*(n)}{\sum_{n=0}^{N-1-k} w(n+k) w^*(n) e^{j\beta n}} \right| = \infty.$$

**Remark 1.** Note that the cyclostationarity concept is introduced here just to explain the performance gain we might expect from the proposed pre-processing technique. Indeed, the damped sinusoids are not cyclostationary signals since the latter are persistent on the time axis and of finite power (not finite energy). As shown by the previous developments, the CS concept is exploited in our work to better mitigate the noise term by using $\beta \neq 0$ in Eq. (4).

**Remark 2.** In the case where the data length is relatively small, one can enhance the signal component by applying the integral of the cyclo-correlation[3] over an interval of the cyclo-frequency:

$$\int_{\beta_0}^{\beta_1} R_\beta(k)\, d\beta = \sum_{m=1}^{L} B_\beta(m) e^{b_m k} \tag{5}$$

where

$$B_\beta(m) = \sum_{l=1}^{L} h_m h_l^* [j(\log(1 - e^{b_m + b_l^* + j\beta_1}) - \log(1 - e^{b_m + b_l^* + j\beta_0})) + \beta_1 - \beta_0]. \tag{6}$$

Note that this integration is equivalent to replacing the unit-norm exponential weight $w(n) = e^{j\beta n}$ by the weight

[1] Note that, in contrast to most existing methods, the noise can be colored and non-Gaussian. The only assumption is its wide sense stationarity.

[2] More precisely, we assume that $e^{\alpha_m N} \ll 1, \ \forall m = 1, \ldots, L$.

[3] Without loss of generality, we omit in the sequel the normalization factor $1/N$ of $R_\beta(k)$.