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A fractional diffusion-wave equation with non-local regularization for image denoising

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ABSTRACT

This paper introduces a novel fractional diffusion-wave equation with non-local regularization for noise removal. Using the fractional time derivative, the model interpolates between the heat diffusion equation and the wave equation, which leads to a mixed behavior of diffusion and wave propagation and thus it can preserve edges in a highly oscillatory region. On the other hand, the usual diffusion is used to reduce the noise whereas the non-local term which exhibits an anti-diffusion effect is used to enhance the image structure. We prove that the proposed model is well-posed, and the stable and convergent numerical scheme is also given in this paper. The experimental results indicate superiority of the proposed model over the baseline diffusion models.

1. Introduction

Partial differential equations (PDEs) based methods for image processing (denoising, restorations, inpainting, segmentation, etc.) have been largely studied in the literature (see [1] and references therein) due to their remarkable advantages in both theory and computation. They allow to directly handle and process visually important geometric features. In addition, they can also effectively simulate several visually meaningful dynamic processes such as linear and nonlinear diffusion, and the information transport mechanism.

In order to preserve the image structures when removing the noise, Perona and Malik [2] proposed a nonlinear equation which replaced isotropic diffusion expressed through a linear heat equation with an anisotropic diffusion. While the backward diffusion of the PM equation results in enhancing the edges, it is an ill-posed process in the sense that it is very sensitive to perturbations in the initial noisy data. Since the work of Perona and Malik, a large number of nonlinear PDEs based anisotropic diffusion models have been proposed [3–5]. In addition, the well-known total variation (TV) model [6] is closely related to the diffusion models and the relations have been studied in the literature [7]. Although these second-order PDEs can achieve a good tradeoff between noise removal and edge preservation, they tend to cause the denoised image to exhibit "staircase effect". In order to overcome this drawback, the high-order PDEs (typically fourth-order PDEs) were adopted in [8–10], but they often lead to the speckle effect.

Recently, fractional-order PDEs have been studied and applied to the image processing and computer vision. Cuesta et al. [11,12] proposed a fractional-order linear integro-differential equation for image denoising as follows:

$$\partial_t^\alpha u(t,x) = \Delta u(t,x), \quad (t,x) \in [0,T] \times \Omega, \tag{1}$$

where Δ denotes Laplacian operator, $\Omega \subset \mathbb{R}^2$ represents the image domain, and ∂_t^α stands for the Riemann–Liouville (R–L) fractional time derivative of order α, $1 < \alpha < 2$. Since the model (1) interpolates a diffusion equation (for $\alpha = 1$) and a wave equation (for $\alpha = 2$), the solution of (1) will satisfy

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intermediate properties, *i.e.*, the maximal diffusion is reached for $\alpha=1$ and there is no diffusion at all for $\alpha=2$. Cao et al. [13] proposed a similar model which replaces the fractional-order derivative by introducing a weight parameter between the first and second order time derivatives. On the other hand, Bai and Feng [14] proposed an anisotropic model with fractional space derivatives, *i.e.*,

$$\partial_t u(t,x) = -(D^\alpha_{x_1})^*(c(\|D^\alpha_x u(t,x)\|^2)D^\alpha_{x_1}) \\ -(D^\alpha_{x_2})^*(c(\|D^\alpha_x u(t,x)\|^2)D^\alpha_{x_2}), \\ t\in[0,T],\quad x=(x_1,x_2)\in\Omega,\quad \alpha\in[1,2], \tag{2}$$

where $D^\alpha_{x_1}$ and $D^\alpha_{x_2}$ denote the fractional spatial derivative of the order α, and $(D^\alpha_\cdot)^*$ represents the adjoint operator of the linear operator $D^\alpha_\cdot$. It is observed that the model (2) leads to an interpolation between the PM model (for $\alpha=1$) and the fourth-order anisotropic diffusion equation [9] (for $\alpha=2$). Thus, it contains the advantages of both methods. In addition, Zhang et al. [15,16] generalized the TV model for image denoising using the Grünwald–Letnikov fractional-order derivative. And Ren et al. [17] proposed the fractional-order TV regularization for image super-resolution. Recently, inspired by the models (1) and (2), Janev et al. [18] proposed a fully fractional anisotropic diffusion (FFAD) equation for noise removal, which contains spatial as well as time fractional derivatives, *i.e.*, $\partial_t u(t,x)$ in (2) is replaced by ${}^cD^\beta_t u(t,x)$ which is the left Caputo time fractional derivative of order $\beta\in[1,2)$. Thus, it can interpolate between the parabolic and the hyperbolic PDE and, at the same time, between the second and fourth order PDE. Although this model can manage to preserve edges and highly oscillatory regions, the anisotropic diffusion is based on the PM model which is actually ill-posed.

Another drawback of the classical PDEs-based methods is that the derivative is a local operator. Recently, the non-local technique have been used very successfully for many image processing tasks [19–22]. These methods exploit the image self-similarities or redundancies to reconstruct the image. However, the process of searching for the similar patches is very time-consuming. In this paper, we attempt to extend the Cuesta's model (1) by introducing a non-local regular term. Note that unlike the non-local methods mentioned above, here the non-local operator is defined by Fourier transform, which dose not require the search process. This non-local operator was first proposed in the field of physics, such as overdriven detonations in gases [23], anomalous diffusion in semiconductor growth [24], the morphodynamics of dunes and drumlins [25,26], etc. Recently, Azerad et al. [27] exploit the anti-diffusion effect of this non-local operator to achieve simultaneous denoising and enhancement of one-dimensional signals. In this paper, we propose a fractional diffusion-wave diffusion with non-local regularization. The usual diffusion smooths the image and removes the noise, while the partial wave-like behavior of the equation and the anti-diffusion of the non-local term guarantee that the edges are preserved. Moreover, the proposed model is actually a linear well-posed equation.

The remainder of this paper is organized as follows: Section 2 introduces the fractional-order derivatives and integrals. Section 3 presents the details of our denoising model. In addition, the well-posedness of the proposed model has been proved in Section 4, and the stable and convergent numerical scheme is given in Section 5. In Section 6, we show experimental validation of our model and compare it, visually and quantitatively, to the baseline diffusion methods. Finally, Section 7 concludes this paper.

2. Fractional calculus

Let us recall the Cauchy's well-known representation of an n-fold integral as a convolution integral

$$J^n f(x) = \frac{1}{(n-1)!}\int_0^x \frac{1}{(x-t)^{1-n}} f(t)\,dt,\quad n\in\mathbb{N}, x\in\mathbb{R}_+, \tag{3}$$

where J^n is the n-fold integral operator with $J^0 f(x)=f(x)$. Replacing n in (3) with $\alpha\in\mathbb{R}_+$, one obtains a definition of a non-integer order integral [28,29], *i.e.*,

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{1}{(x-t)^{1-\alpha}} f(t)\,dt,\quad x\in\mathbb{R}_+, \tag{4}$$

where $\Gamma(\alpha)$ is the Euler's continuous gamma function, which satisfies $\Gamma(\alpha)=(\alpha-1)!$ if α is an integer. The non-integer order derivatives can simplest be defined as concatenation of integer order differentiation and fractional integration, *i.e.*,

$$D^\alpha f(x) = D^n J^{n-\alpha} f(x) \quad \text{or} \quad {}^cD^\alpha f(x) = J^{n-\alpha} D^n f(x), \tag{5}$$

where n is the integer satisfying $\alpha\le n<\alpha+1$ and D^n, $n\in\mathbb{N}$, is the n-fold differential operator with $D^0 f(x)=f(x)$. The operator D^α is called the Riemann–Liouville (R–L) differential operator of order α, while the operator ${}^cD^\alpha$ is named Caputo differential operator of the order α [28,29].

3. Proposed model

As mentioned above, the model (1) [11,12] is a well-posed equation which interpolates a diffusion equation and a wave equation, but it is based on the heat equation which performs an isotropic diffusion. Although the model (2) [14] interpolates between the second and fourth order anisotropic diffusion equation and the Janev's model [18] can also interpolate between the parabolic and the hyperbolic PDE at the same time, they are based on the PM equation which is actually ill-posed. To overcome their drawbacks, we propose a fractional diffusion-wave equation with the non-local regular term

$$\begin{cases} {}^cD^\alpha_t u(t,x) = \Delta u(t,x) + \lambda g_\beta[u(t,\cdot)](x), & (t,x)\in[0,T]\times\Omega, \\ u(0,x)=u_0(x), & x\in\Omega, \\ \partial_t u(0,x)=0, & x\in\Omega, \\ \dfrac{\partial u}{\partial \vec{n}}(t,x)=0, & (t,x)\in(0,T)\times\partial\Omega, \end{cases} \tag{6}$$

where $u_0\in L_2(\Omega)$ represents the noisy image, $\vec{n}$ denotes the exterior normal to the boundary $\partial\Omega$, λ is a positive parameter, and ${}^cD^\alpha_t$ denotes the partial Caputo differential operator of order $\alpha\in[1,2)$ with respect to time t. We use Caputo differential operator because of its advantage with regards to the initial conditions, *i.e.*, the fact that the initial and boundary conditions retain the form given in (6).

In (6), $g_\beta[\varphi]$ is the non-local operator, also called Lévy operator, defined by the symbol $|\xi|^\beta$, $1<\beta<2$. More