Signal Processing 90 (2010) 2083–2086

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

Fast communication

Iterative quadratic maximum likelihood based estimator for a biased sinusoid

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ARTICLE INFO

Article history:
Received 18 August 2009
Received in revised form
12 November 2009
Accepted 17 November 2009
Available online 26 November 2009

Keywords:
Parameter estimation
Sinusoid
Offset
Linear prediction
Maximum likelihood

ABSTRACT

The problem of parameter estimation of a single sinusoid with unknown offset in additive Gaussian noise is addressed. After deriving the linear prediction property of the noise-free signal, the maximum likelihood estimator for the frequency parameter is developed. The optimum estimator is relaxed according to the iterative quadratic maximum likelihood technique. The remaining parameters are then solved in a linear least squares manner. Theoretical variance expression of the frequency estimate based on high signal-to-noise ratio assumption is also derived. Simulation results show that the proposed algorithm can give optimum estimation performance and is superior to the nonlinear least squares approach.

1. Introduction

Parameter estimation of sinusoidal signals in additive noise has been an active research topic [1–5] because of its numerous application areas in power delivery [6], signal processing [7], digital communications [8], instrumentation and measurement [9] and so on. In this paper, we tackle the parameter estimation problem for a real biased sinusoid. The observed signal, which is also known as the four-parameter sine wave model [9,10], is

$$x(n) = s(n) + q(n), \quad n = 1, 2, \ldots, N \tag{1}$$

where

$$s(n) = A\cos(\omega n + \phi) + B \tag{2}$$

where $A > 0$, $\omega \in (0, \pi)$, $\phi \in [0, 2\pi)$ and B are deterministic but unknown constants which denote the tone amplitude, frequency, phase and offset, respectively, while $q(n)$ is the additive zero-mean Gaussian noise. Our objective is to estimate A, ω, ϕ and B, from the N discrete-time noisy measurements $\{x(n)\}$.

Although there are numerous sinusoidal parameter estimation schemes in the literature such as maximum likelihood (ML), nonlinear least squares (NLS) [4,10], iterative quadratic maximum likelihood (IQML) [11,12], linear prediction (LP) [2], most of them assume $B = 0$. In fact, it is of interest to estimate the non-zero offset or DC value as well [9,10]. In this paper, we contribute to the development of an accurate and computationally attractive parameter estimation approach for a single tone with non-zero offset. We first derive the LP property of (2) and then produce the ML estimator for the frequency parameter. As the ML cost function is multi-modal, IQML-based relaxation is utilized to yield a simple iterative algorithm. The parameters of amplitude, phase and offset are then obtained in a linear least squares (LLS) manner. Furthermore, the variance of the frequency estimate in high signal-to-noise ratio (SNR) conditions is theoretically analyzed. The effectiveness of the proposed scheme is demonstrated by comparing with the NLS approach and Cramér–Rao lower bound (CRLB) [10].

2. Algorithm development

It is well known that when $B = 0$, $s(n)$ obeys the LP property of $s(n) + s(n-2) = 2\cos(\omega)s(n-1)$. For nonzero B,

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0165-1684/$ - see front matter
doi:10.1016/j.sigpro.2009.11.025

its extension is

$$s(n)+s(n-2)-2B=2\cos(\omega)(s(n-1)-B)$$
$$\Rightarrow s(n)+s(n-2)-2\cos(\omega)s(n-1)=2B(1-\cos(\omega)) \quad (3)$$

where we note that $2B(1-\cos(\omega))$ is independent of the index n. Substituting n with $n-1$ in (3) yields another equality and equating with (3), we have

$$s(n)+s(n-2)-2\cos(\omega)s(n-1)=s(n-1)+s(n-3)-2\cos(\omega)s(n-2)$$
$$\Rightarrow s(n)-s(n-3)-(2\cos(\omega)+1)(s(n-1)-s(n-2))=0 \quad (4)$$

which is the LP property of single real tone with offset. As in conventional sinusoidal parameter estimation, finding ω is the first and crucial step because it is a nonlinear function in the received data sequence. The remaining parameters, namely, A, ϕ and B can then be estimated in a more straightforward manner after its determination.

Let $\rho=2\cos(\omega)+1$ and define $\mathbf{x}_i=[x(i),x(i+1),\ldots,x(i+N-4)]^T$, $i=1,2,3,4$, where ^{T} denotes the transpose operator. Using (4) and following the development in [13], it is shown that the ML estimate for ρ, denoted by $\hat{\rho}$, in the presence of Gaussian noise can be determined from the following minimization problem:

$$\hat{\rho}=\arg\min_{\tilde{\rho}}(\mathbf{x}_4-\mathbf{x}_1-\tilde{\rho}(\mathbf{x}_3-\mathbf{x}_2))^T\boldsymbol{\Sigma}(\tilde{\rho})^{-1}(\mathbf{x}_4-\mathbf{x}_1-\tilde{\rho}(\mathbf{x}_3-\mathbf{x}_2)) \quad (5)$$

where $\tilde{\rho}$ is the optimization variable for ρ and $^{-1}$ represents the matrix inverse. The covariance matrix $\boldsymbol{\Sigma}(\tilde{\rho})$ is also a function of $\tilde{\rho}$ and has the form of $E\{\mathbf{p}\mathbf{p}^T\}$ with $\mathbf{p}=[p(1),p(2),\ldots,p(N-3)]^T$ whose element is $p(n)=q(n+3)-q(n)-\tilde{\rho}(q(n+2)-q(n+1))$, $n=1,2,\ldots,N-3$. For zero-mean white Gaussian noise, $\boldsymbol{\Sigma}(\tilde{\rho})$ is expressed as

$$\boldsymbol{\Sigma}\tilde{\rho}=\text{Toeplitz}([2(\tilde{\rho}^2+1)\ \ -\tilde{\rho}^2-2\tilde{\rho}\ \ 2\tilde{\rho}\ \ -1\ \ 0\cdots0])\sigma^2$$
$$=\begin{bmatrix} 2(\tilde{\rho}^2+1) & -\tilde{\rho}^2-2\tilde{\rho} & 2\tilde{\rho} & -1 & 0 & \cdots & 0 \\ -\tilde{\rho}^2-2\tilde{\rho} & (2\tilde{\rho}^2+1) & -\tilde{\rho}^2-2\tilde{\rho} & 2\tilde{\rho} & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\tilde{\rho}^2-2\tilde{\rho} & 2(\tilde{\rho}^2+1) \end{bmatrix}\sigma^2 \quad (6)$$

It is noteworthy that this ML estimator can be extended to more general Gaussian process of $q(n)$ as long as $\boldsymbol{\Sigma}(\tilde{\rho})$ is known up to a scalar. However, the objective function in (5) is multi-modal and thus there is no guarantee that the globally optimum point can be obtained.

Utilizing the idea of the IQML technique [11,12], we relax (5) into a quadratic function by considering $\boldsymbol{\Sigma}(\tilde{\rho})$ is independent of $\tilde{\rho}$ so that global optimization is attained and the estimate of ρ, denoted by $\hat{\rho}$, is easily computed as

$$\hat{\rho}=\frac{(\mathbf{x}_3-\mathbf{x}_2)^T\boldsymbol{\Sigma}(\hat{\rho})^{-1}(\mathbf{x}_4-\mathbf{x}_1)}{(\mathbf{x}_3-\mathbf{x}_2)^T\boldsymbol{\Sigma}(\hat{\rho})^{-1}(\mathbf{x}_3-\mathbf{x}_2)} \quad (7)$$

Note that σ^2 of $\boldsymbol{\Sigma}(\tilde{\rho})$ in the numerator and denominator cancel each other and hence its value is not required to be known. We iterate (7) with an initial guess of $\boldsymbol{\Sigma}(\hat{\rho})$ while the estimated ρ is then employed to update $\boldsymbol{\Sigma}(\hat{\rho})$. The iterative procedure of the proposed estimator for ω is summarized as follows:

(i) Set $\boldsymbol{\Sigma}(\hat{\rho})=\mathbf{I}_{N-3}$ which is the $(N-3)\times(N-3)$ identity matrix.
(ii) Compute $\hat{\rho}$ using (7).
(iii) Use $\hat{\rho}$ to construct $\boldsymbol{\Sigma}(\hat{\rho})$ of (6).
(iv) Repeat Steps (ii) and (iii) until a stopping criterion is reached. In this study, we terminate for a fixed number of iterations, although the procedure can also be stopped when the absolute difference between successive estimates of ρ is less than ε which is a sufficiently small positive constant.
(v) Compute $\hat{\omega}$ using

$$\hat{\omega}=\cos^{-1}\left(\frac{\hat{\rho}-1}{2}\right) \quad (8)$$

Employing $\hat{\omega}$, the estimates of A, ϕ and B, denoted by $\hat{A}$, $\hat{\phi}$ and $\hat{B}$, respectively, are obtained by minimizing the following LLS cost function:

$$(\boldsymbol{\Xi}\boldsymbol{\kappa}-\mathbf{x})^T(\boldsymbol{\Xi}\boldsymbol{\kappa}-\mathbf{x}) \quad (9)$$

where

$$\boldsymbol{\Xi}=\begin{bmatrix} \cos(\hat{\omega}) & \cos(2\hat{\omega}) & \cdots & \cos(N\hat{\omega}) \\ -\sin(\hat{\omega}) & -\sin(2\hat{\omega}) & \cdots & -\sin(N\hat{\omega}) \\ 1 & 1 & \cdots & 1 \end{bmatrix}^T$$

$$\boldsymbol{\kappa}=[A\cos(\phi)\ \ A\sin(\phi)\ \ B]^T$$

and

$$\mathbf{x}=[x(1)\ \ x(2)\ \ \cdots\ \ x(N)]^T$$

From (9), the LLS estimate of $\boldsymbol{\kappa}$ is

$$\hat{\boldsymbol{\kappa}}=[[\hat{\boldsymbol{\kappa}}]_1\ \ [\hat{\boldsymbol{\kappa}}]_2\ \ [\hat{\boldsymbol{\kappa}}]_3]^T=(\boldsymbol{\Xi}^T\boldsymbol{\Xi})^{-1}\boldsymbol{\Xi}^T\mathbf{x} \quad (10)$$

which gives

$$\hat{A}=\sqrt{[\hat{\boldsymbol{\kappa}}]_1^2+[\hat{\boldsymbol{\kappa}}]_2^2} \quad (11)$$

$$\hat{\phi}=\tan^{-1}\left(\frac{[\hat{\boldsymbol{\kappa}}]_2}{[\hat{\boldsymbol{\kappa}}]_1}\right) \quad (12)$$

and

$$\hat{B}=[\hat{\boldsymbol{\kappa}}]_3 \quad (13)$$

3. Variance analysis

In this section, we analyze the variance of $\hat{\omega}$ based on high SNR assumption. Let $\mathbf{y}=\mathbf{x}_3-\mathbf{x}_2=\overline{\mathbf{y}}+\Delta\mathbf{y}$ and $\mathbf{z}=\mathbf{x}_4-\mathbf{x}_1=\overline{\mathbf{z}}+\Delta\mathbf{z}$ where $\overline{\mathbf{g}}$ and $\Delta\mathbf{g}$ are the noise-free version and perturbation of $\mathbf{g}$, respectively. Upon parameter convergence, (7) implies

$$f(\hat{\rho})=\mathbf{y}^T\boldsymbol{\Sigma}(\hat{\rho})^{-1}(\mathbf{y}\hat{\rho}-\mathbf{z})=0 \quad (14)$$

The $f(\hat{\rho})$ can be linearized using Taylor's series as

$$0=f(\hat{\rho})\approx f(\rho)+f'(\rho)\Delta\rho \quad (15)$$

where $f(\rho)=y^T\boldsymbol{\Sigma}(\rho)^{-1}(\mathbf{y}\rho-\mathbf{z})$, $f'(\rho)=\mathbf{y}^T\boldsymbol{\Sigma}(\rho)^{-1}\mathbf{y}+\mathbf{y}^T\boldsymbol{\Sigma}(\rho)^{-1}\boldsymbol{\Sigma}(\rho)'\boldsymbol{\Sigma}(\rho)^{-1}(\mathbf{z}-\mathbf{y}\rho)$ and $\Delta\rho=\hat{\rho}-\rho$.

Here, $f'(\rho)$ and $\boldsymbol{\Sigma}(\rho)'$ are the first derivatives of $f(\rho)$ and $\boldsymbol{\Sigma}(\rho)$, respectively. Although $f(\hat{\rho})$ has multiple roots and we cannot guarantee that our obtained root from the iterative procedure corresponds to ρ, it is expected that $\hat{\rho}$