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Design of fractional delay filter using discrete Fourier transform interpolation method

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ABSTRACT

In this paper, the design of fractional delay FIR filter is investigated. First, the interpolation formula of a discrete-time sequence is derived by using discrete Fourier transform (DFT). Then, the DFT-based interpolation formula is applied to design fractional delay FIR filter by using suitable index mapping. The filter coefficients are easily computed because a closed-form design is obtained. Next, design examples are demonstrated to show that the proposed DFT method has a smaller design error than those of the conventional Lagrange and window fractional delay FIR filters when using the same design parameters. Finally, the designed DFT-based fractional delay FIR filter is applied to design a digital differentiator and half-band filter.

1. Introduction

In many signal processing applications, there is a need for a delay which is a fraction of the sampling period. These applications include beam steering of antenna arrays, time adjustment in digital receivers, modeling of music instruments, speech coding and synthesis, comb filter design and analog digital conversion, etc. [1–6]. An excellent survey of fractional delay filter design is presented in tutorial paper [1]. Recently, the relationships among fractional delay, differentiator, Nyquist filter, lowpass filter and diamond-shaped filter have been established such that fractional delay becomes a versatile building block in the design of these practical filters [7–9]. The ideal frequency response of a fractional delay filter is given by

$$D(\omega) = e^{-j\omega(I+d)} \tag{1}$$

where I is a positive integer and d is a fractional number in the interval [0,1). The transfer function of the FIR filter of length N used to approximate this specification is given by

$$H(z) = \sum_{r=0}^{N-1} h(r) z^{-r} \tag{2}$$

So far, several methods of designing an FIR filter $H(z)$ to fit fractional delay specification $D(\omega)$ as closely as possible have been developed. Two typical approaches are Lagrange interpolation method and window method [1].

On the other hand, the discrete Fourier transform (DFT) is defined as follows:

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \le k \le N-1$$

$$x(n) = \sum_{k=0}^{N-1} \frac{1}{N} X(k) W_N^{-kn}, 0 \le n \le N-1 \tag{3}$$

where $W_N = e^{-j2\pi/N}$. It is well known that the DFT is a useful tool for us to design digital filters. One typical example is the equiripple FIR filter design by the fast Fourier transform (FFT) algorithm [10]. Moreover, the zero padding in the frequency domain provides interpolation

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in the time domain, so we should be able to use zero padding in the high frequency range of the DFT domain as a means of interpolating finite-duration, discrete-time signals. The details can be found in the literature [11–17]. So far, the DFT interpolation and fractional delay filter design are two independent research topics in signal processing area. The relation between them has not been investigated. The purpose of this paper is to study this relation such that DFT interpolation concept can be directly used to design fractional delay FIR filters. The main advantage is that the filter coefficients $h(r)$ are easily computed because a closed-form design is obtained. This paper is organized as follows. In Section 2, the interpolation formulas for odd and even length N are derived. In Section 3, the interpolation formula is applied to design fractional delay FIR filter. There are two kinds of fractional delay filters to be designed. One concerns the case of even-length N, the other is the case of odd-length N. In Section 4, numerical examples are included in order to compare the filters designed using the proposed DFT method with conventional Lagrange and window fractional delay FIR filters. In Section 5, the designed DFT-based fractional delay FIR filter is also applied to design a digital differentiator and half-band filter. Finally, a conclusion is made.

2. DFT interpolation formula

In this section, the DFT interpolation formula with even-length sequence is first derived. Then, the odd-length case is discussed. Given DFT $X(k)$ of an even-length real-valued sequence $x(n)$, let us define the zero-padded DFT as

$$X_d(k)=\begin{cases} LX(k), & 0\le k\le \frac{N}{2}-1 \\ \frac{1}{2}LX(\frac{N}{2}), & k=\frac{N}{2} \\ 0, & \frac{N}{2}+1\le k\le M-\frac{N}{2}-1 \\ \frac{1}{2}LX(\frac{N}{2}), & k=M-\frac{N}{2} \\ LX(k-M+N), & M-\frac{N}{2}+1\le k\le M-1 \end{cases} \tag{4}$$

Also, we assume M is an integer multiple of N, say M=LN, where L is called interpolation factor. The above DFT has zero values in the high frequency range and satisfies the following conjugate symmetry condition:

$$X_d(M-k)=X_d(k)^*, \quad 1\le k\le \frac{N}{2} \tag{5}$$

Now, the interpolated sequence $x_d(n)$ is chosen as the length-M inverse DFT of $X_d(k)$, i.e.,

$$\begin{aligned} x_d(n)&=\frac{1}{M}\sum_{k=0}^{M-1}X_d(k)W_M^{-kn} \\ &=\frac{1}{N}\Big(\sum_{k=0}^{N/2-1}X(k)W_M^{-kn}+\sum_{k=M-N/2+1}^{M-1}X(k)W_M^{-kn}\Big) \\ &\quad+\frac{1}{2N}\left(X\left(\frac{N}{2}\right)W_M^{-(N/2)n}+X\left(\frac{N}{2}\right)W_M^{-(M-N/2)n}\right) \end{aligned} \tag{6}$$

Let $k'=M-k$ and using the equality $W_M^{-Mn}=1$ and Eq. (5), the second term in Eq. (6) can be written as

$$\begin{aligned} \sum_{k=M-N/2+1}^{M-1}X(k)W_M^{-kn}&=\sum_{k'=1}^{N/2-1}X(M-k')W_M^{-(M-k')n} \\ &=\sum_{k'=1}^{N/2-1}X(k')^*W_M^{k'n} \end{aligned} \tag{7}$$

Substituting Eq. (7) into Eq. (6) and using the equality $W_M^{-(M-N/2)n}=W_M^{Nn/2}$, we have

$$\begin{aligned} x_d(n)&=\frac{1}{N}[X(0)+\sum_{k=1}^{N/2-1}(X(k)W_M^{-kn}+X(k)^*W_M^{kn})] \\ &\quad+\frac{1}{2N}X\left(\frac{N}{2}\right)(W_M^{-(N/2)n}+W_M^{(N/2)n}) \end{aligned} \tag{8}$$

Using DFT defined in Eq. (3), the first term in Eq. (8) can be written as

$$\begin{aligned} R_1&=\frac{1}{N}\sum_{m=0}^{N-1}x(m)[1+\sum_{k=1}^{N/2-1}(W_N^{mk}W_M^{-kn}+W_N^{-mk}W_M^{kn})] \\ &=\frac{1}{N}\sum_{m=0}^{N-1}x(m)\left[1+2\sum_{k=1}^{N/2-1}\cos\left(\frac{-2\pi mk}{N}+\frac{2\pi nk}{M}\right)\right] \end{aligned} \tag{9}$$

Moreover, using the equality $W_N^{N/2}=-1$, the second term in Eq. (8) can be written as

$$\begin{aligned} R_2&=\frac{1}{2N}X\left(\frac{N}{2}\right)\left(W_M^{-(N/2)n}+W_M^{(N/2)n}\right) \\ &=\frac{1}{2N}\sum_{m=0}^{N-1}x(m)W_N^{(N/2)m}2\cos\left(\frac{n\pi N}{M}\right) \\ &=\frac{1}{N}\sum_{m=0}^{N-1}(-1)^m x(m)\cos\left(\frac{n\pi N}{M}\right) \end{aligned} \tag{10}$$

Substituting Eq. (9) and Eq. (10) into Eq. (8), the sequence $x_d(n)$ becomes the form:

$$\begin{aligned} x_d(n)&=R_1+R_2 \\ &=\frac{1}{N}\sum_{m=0}^{N-1}x(m)\left\{1+2\sum_{k=1}^{N/2-1}\cos\left(\frac{-2\pi mk}{N}+\frac{2\pi nk}{M}\right)\right. \\ &\quad\left.+(-1)^m\cos\left(\frac{n\pi N}{M}\right)\right\} \end{aligned} \tag{11}$$

It can be shown that the above interpolator satisfies the following equality:

$$x_d(iL)=x(i) \tag{12}$$

Because the sequence $x_d(n)$ is the interpolated version of the sequence $x(n)$ with factor L, we have the following relation:

$$x_d(iL+p)\approx x\left(i+\frac{p}{L}\right) \tag{13}$$

for $0\le p\le L-1$ and $0\le i\le N-1$. Combining Eqs. (11) and (13), we have

$$\begin{aligned} x\left(i+\frac{p}{L}\right)&\approx x_d(iL+p) \\ &=\frac{1}{N}\sum_{m=0}^{N-1}x(m)\left\{1+2\sum_{k=1}^{N/2-1}\cos\left(\frac{2\pi(i+\frac{p}{L}-m)k}{N}\right)\right. \end{aligned}$$