Signal Processing 89 (2009) 1483–1488

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

Review

Fibonacci sequence, golden section, Kalman filter and optimal control

A. Benavoli [a,*], L. Chisci [b], A. Farina [c]

[a] Istituto Dalle Molle di Studi sull'Intelligenza Artificiale Manno, Switzerland
[b] DSI, Università di Firenze, Firenze, Italy
[c] Engineering Division, SELEX - Sistemi Integrati S.p.A., Rome, Italy

ARTICLE INFO

Article history:
Received 18 November 2008
Received in revised form
30 January 2009
Accepted 6 February 2009
Available online 20 February 2009

Keywords:
Kalman filter
Linear quadratic control
Golden section/ratio
Fibonacci sequence
Random walk

ABSTRACT

A connection between the Kalman filter and the Fibonacci sequence is developed. More precisely it is shown that, for a scalar random walk system in which the two noise sources (process and measurement noise) have equal variance, the Kalman filter's estimate turns out to be a convex linear combination of the *a priori* estimate and of the measurements with coefficients suitably related to the Fibonacci numbers. It is also shown how, in this case, the steady-state Kalman gain as well as the predicted and filtered covariances are related to the golden ratio $\phi = (\sqrt{5}+1)/2$. Furthermore, it is shown that, for a generic scalar system, there exist values of its key parameters (i.e. system dynamics and ratio of process-to-measurement noise variances) for which the previous connection is preserved. Finally, by exploiting the duality principle between control and estimation, similar connections with the linear quadratic control problem are outlined.

Contents

1. Introduction

The Fibonacci numbers and the golden section are ubiquitous in nature and art as well as in science and engineering. Is there any link between the Nautilus shell, the layout of the sunflower's seeds and the proportions of the Botticelli's Venus? The answer to this question is the irrational number $\phi = 1.618033988749895\ldots$, known as

* Corresponding author.
E-mail addresses: benavoli@gmail.com (A. Benavoli), chisci@dsi.unifi.it (L. Chisci), afarina@selex-si.com (A. Farina).

0165-1684/$ - see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.sigpro.2009.02.003

golden ratio [1], which is perhaps the first irrational number ever discovered. The number ϕ has interesting properties. Its square is equal to $\phi+1$, its reciprocal (called *golden section*) is equal to $\phi-1$ and, further, ϕ is intimately connected to the sequence of Fibonacci numbers $\{0,1,1,2,3,5,8,13,21,34,\ldots\}$. The Fibonacci sequence, starting from 0 and 1, is defined by recurrence by taking each subsequent number as the sum of the two previous ones. It can easily be checked that the ratio between a number of the sequence and the previous one converges to the golden ratio.

The goal of this paper is to show the existence of a connection between the Fibonacci sequence and the golden section/ratio on one side and the Kalman filter on the other side. By applying the duality principle between estimation and control, a connection with the linear quadratic (LQ) control problem is also derived. Before proceeding, it should be pointed out that this connection must be regarded as a mere mathematical curiosity which, however, makes the history of ϕ even more fascinating.

The rest of the paper is organized as follows. Section 2 reviews some historical notes; Section 3 displays the connection between the Fibonacci sequence and the Kalman filter and, finally, Section 4 lists the conclusions.

2. Historical notes

2.1. Golden section

The first mathematical definition of *golden ratio* traces back to the famous Greek mathematician Euclid who, in the 3rd century BC, introduced it [2] to solve a geometrical problem called the problem of *division of a line segment in extreme and mean ratio*. The essence of the problem is the following. A line segment AB must be divided with a point C into two parts so that the ratio between the longer part CB and the shorter one AC is equal to the ratio between the whole line segment AB and the longer part CB, i.e.:

$$\frac{AB}{CB}=\frac{CB}{AC} \tag{2.1}$$

Exploiting the relationship $AB = AC + CB$, Eq. (2.1) can be written in the following form:

$$x=\frac{CB}{AC}=\frac{AB}{CB}=\frac{AC+CB}{CB}=1+\frac{AC}{CB}=1+\frac{1}{x} \tag{2.2}$$

Hence, the equation to calculate the ratio x is

$$x^2-x-1=0 \tag{2.3}$$

The positive root of Eq. (2.3) is the solution of the problem of *division of a line segment in extreme and mean ratio*. This solution is just the *golden ratio*:

$$\phi=\frac{1+\sqrt{5}}{2}\approx 1.618 \tag{2.4}$$

The discovery of the number ϕ, however, cannot be attributed to Euclid but probably to Pythagoras or to Hippasus of Metapontum (a disciple of Pythagoras) in the 5th century BC [1]. In fact, the ancient philosopher Pythagoras chose the pentagram (a regular star with five vertices) as the symbol of the secret fraternity of which he was both leader and founder and the construction of a pentagram is based on the golden ratio. The pentagram can be seen as a geometric shape consisting of five straight lines arranged as a star with five points. The intersection of the lines naturally divides each line into three parts. The smaller part (which forms the pentagon inside the star) is proportional to the longer parts (which form the points of the star) by a ratio of $1:\phi$. For the connection between Pythagoreans and the pentagram it is reasonable to believe that the Pythagoreans were the first to discover ϕ and so the first to discover the existence of irrational numbers. The mathematical history of ϕ went on and today many interesting things about this number are known. A list of the mathematical properties of ϕ is given below:

- ϕ can be expressed as a continuous fraction with only 1:

$$\phi=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}$$

- ϕ can be expressed as a continuous square root

$$\phi=\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}$$

- Any power of ϕ is equal to the sum of the two immediately preceding powers:

$$\phi^n=\phi^{n-1}+\phi^{n-2}$$

It is worth pointing out that ϕ has many other mathematical and geometrical properties, but the most interesting is certainly the connection with the Fibonacci sequence.

2.2. Leonardo Pisano Fibonacci

Leonardo Pisano (1170–1250), better known by his nickname Fibonacci, was born in Italy but was educated in North Africa where his father, Guilielmo, held a diplomatic post [3]. His father's job was to represent the merchants of the Republic of Pisa who were trading in Bugia, later called Bougie and now called Bejaia (Algeria). Fibonacci was taught mathematics in Bugia and travelled widely with his father and recognized the enormous advantages of the mathematical systems used in the countries they visited (Fig. 1).

Fibonacci ended his travels around the year 1200 and at that time he returned to Pisa [3]. There he wrote a number of important texts which played an important role in reviving ancient mathematical skills and he made significant contributions of his own.

The most important book of Fibonacci, Liber abaci (the book of calculations) [4], was published in 1202 after