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Fast communication

Route to chaos in a third-order phase-locked loop network

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ABSTRACT

Phase-locked loops (PLLs) are widely used in applications related to control systems and telecommunication networks. Here we show that a single-chain master–slave network of third-order PLLs can exhibit stationary, periodic and chaotic behaviors, when the value of a single parameter is varied. Hopf, period-doubling and saddle–saddle bifurcations are found. Chaos appears in dissipative and non-dissipative conditions. Thus, chaotic behaviors with distinct dynamical features can be generated. A way of encoding binary messages using such a chaos-based communication system is suggested.

1. Introduction

Phase-locked loop (PLL) is an electronic device designed to extract time signals from transmission channels. This device has been extensively employed in applications requiring automatic control of frequency with the aim of obtaining synchronism, such as in computers, modems, motors, radars, radio and television receivers, telecommunication networks, etc. (e.g. [1,2]). It is a closed loop composed by three elements: a phase detector (PD), a low-pass filter (LPF) and a voltage controlled oscillator (VCO), as illustrated in Fig. 1.

Consider a single-chain master–slave telecommunication network, where each node sends signals to a unique neighboring node. Let $\theta_{i(j)}(t)$ be the phase of the input signal and $\theta_{o(j)}(t)$ the phase of the output signal of the j-th PLL. The role of j-th PLL is to synchronize the signal $v_{o(j)}(t)$ generated by its own VCO with the signal $v_{i(j)} = v_{o(j-1)}(t)$ provided by VCO of the $(j-1)$-th PLL $(j = 1, 2, \ldots)$.

Assume that:

$$v_{o(j)}(t) = V_{o(j)} \cos\left[\omega_0 t + \theta_{o(j)}(t) + (j-1)\frac{\pi}{2}\right] \tag{1}$$

for $j = 0, 1, 2, \ldots$. Thus, the output signal of every VCO has periodic form with central frequency ω_0 and amplitude $V_{o(j)} > 0$. The index $j = 0$ labels the master clock.

The adjustable phase of the output signal of j-th PLL is $\theta_{o(j)}(t)$ and it depends on the time-varying phase $\theta_{i(j)}(t)$ of the input signal. A synchronous solution corresponds to the phase errors defined by $\phi_j(t) \equiv \theta_{i(j)}(t) - \theta_{o(j)}(t) = \theta_{o(j-1)}(t) - \theta_{o(j)}(t)$ $(j = 1, 2, \ldots)$ assuming constant values or, equivalently, the frequency errors $d\phi_j(t)/dt \equiv w_j(t) = d\theta_{i(j)}(t)/dt - d\theta_{o(j)}(t)/dt = d\theta_{o(j-1)}(t)/dt - d\theta_{o(j)}(t)/dt$ vanishing (e.g. [3–6]).

We consider that the input–output relation concerning the LPF of the j-th PLL is described by the second-order differential equation:

$$\frac{d^2 v_{c(j)}(t)}{dt^2} + k_j \frac{dv_{c(j)}(t)}{dt} + v_{c(j)}(t) = \frac{dv_{d(j)}(t)}{dt} + v_{d(j)}(t) \tag{2}$$

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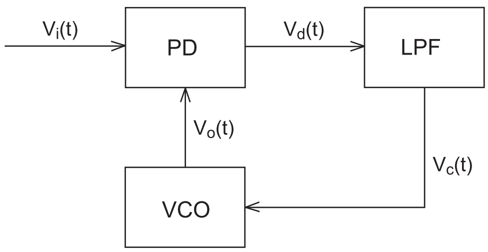

Fig. 1. Block diagram of a PLL. The PLL input signal is represented by $v_i(t)$, the VCO output signal by $v_o(t)$, the PD output signal by $v_d(t)$ and the LPF output signal by $v_c(t)$.

where $v_{d(j)}(t)$ is the input and $v_{c(j)}(t)$ is the output of the LPF, and $k_j \geqslant 0$. PLLs with similar filters were already studied (e.g. [4,7,8]).

Here all PLLs use signal multiplier as PD; therefore, the PD output $v_{d(j)}(t)$ is given by:

$$v_{d(j)}(t) = k_{d(j)} v_{i(j)}(t) v_{o(j)}(t) = k_{d(j)} v_{o(j-1)}(t) v_{o(j)}(t) \tag{3}$$

where $k_{d(j)} > 0$ is the PD gain of the j-th PLL.

The VCO output phase $\theta_{o(j)}(t)$ is controlled by the signal $v_{c(j)}(t)$ according to:

$$\frac{d\theta_{o(j)}(t)}{dt} = k_{v(j)} v_{c(j)}(t) \tag{4}$$

where $k_{v(j)} > 0$ is the VCO gain of the j-th PLL.

It is a common approximation to consider that the second-harmonic appearing in $v_{d(j)}(t)$ will be cut out by the filter (for a discussion, see [3,5]). Thus, the expression for $v_{d(j)}(t)$ can be reduced to:

$$v_{d(j)}(t) \simeq \frac{k_{d(j)} V_{o(j)} V_{o(j-1)}}{2} \sin \phi_j(t) \tag{5}$$

By combining the expressions (1)–(5), the dynamics of the j-th PLL is described by the following nonlinear ordinary differential equation:

$$\frac{d^3\phi_j(t)}{dt^3} + k_j \frac{d^2\phi_j(t)}{dt^2} + (1 + \mu_j \cos \phi_j(t)) \frac{d\phi_j}{dt} + \mu_j \sin \phi_j(t) = \frac{d^3\theta_{i(j)}(t)}{dt^3} + k_j \frac{d^2\theta_{i(j)}(t)}{dt^2} + \frac{d\theta_{i(j)}(t)}{dt} \equiv g_j(t) \tag{6}$$

where $\mu_j \equiv (V_{o(j)} V_{o(j-1)} k_{d(j)} k_{v(j)})/2 > 0$ is called PLL gain.

Since 1980s chaotic circuits (e.g. [9,10]) have been theoretically analyzed and physically built in order to be used in applications involving cryptography (e.g. [11,12]), image processing (e.g. [13]), modulation (e.g. [11,12]), network synchronization (e.g. [14]), pseudo-random number generation (e.g. [11,12]), etc. Here we analytically and numerically investigate the asymptotical solutions of the network described by Eq. (6) and propose a way of encoding binary messages using the chaotic behaviors appearing in such a network. Analyses for first-order (e.g. [15–17]), second-order (e.g. [18,19]), and different third-order PLL networks (e.g. [4,6,8,20]) can be found in the literature.

2. Analysis

Firstly, consider the case where there are two nodes; that is, only one slave ($j = 1$) linked to the master clock ($j = 0$). Assume that the master phase $\theta_{o(0)}(t)$ presents a linear variation with the time, that is: $\theta_{o(0)}(t) = \Omega t + c$, with $\Omega \geqslant 0$ and $c =$ constant. Observe that when $\theta_{o(0)}(t) \equiv \theta_{i(1)}(t)$ varies as a ramp input ($\Omega \neq 0$), then $g_1(t) = \Omega$ becomes a step input.

The third-order differential Eq. (6) for $\phi_1(t) \equiv \theta_{o(0)}(t) - \theta_{o(1)}(t)$ can be rewritten as the following three first-order differential equations:

$$\frac{d\phi_1(t)}{dt} \equiv w_1(t) \equiv f_1(\phi_1, w_1, a_1)$$
$$\frac{dw_1(t)}{dt} \equiv a_1(t) \equiv f_2(\phi_1, w_1, a_1)$$

$$\frac{da_1(t)}{dt} = -k_1 a_1(t) - (1 + \mu_1 \cos \phi_1(t)) w_1(t) - \mu_1 \sin \phi_1(t) + \Omega \equiv f_3(\phi_1, w_1, a_1) \tag{7}$$

Notice that $\vec{\nabla} \cdot \vec{f}(\phi_1, w_1, a_1) = -k_1$, where $\vec{f} = (f_1, f_2, f_3)$. Thus, the divergent of the vector field $\vec{f}$ related to the system (7) is negative for $k_1 > 0$, implying that the system is dissipative (which means that volumes in the state space $\phi_1 \times w_1 \times a_1$ contract along the flow). For $k_1 = 0$ such a divergent is null; hence, this system is conservative (which means that volumes in the state space are preserved).

In the PLL jargon, the capture range is defined as the set of values of the velocity Ω such that the closed loop is able of reaching a synchronous state. This state corresponds to a stationary solution with $\phi_1(t) = \phi_1^* =$ constant, $w_1(t) = w_1^* = 0$, $a_1(t) = a_1^* = 0$ and is represented by the equilibrium point $(\phi_1^*, 0, 0)$ in the state space.

The nonlinear system (7) presents two equilibrium points: a point with $\phi_{1a}^* = \arcsin(\Omega/\mu_1)$ $(0 \leqslant \phi_{1a}^* \leqslant \pi/2)$ and another point with $\phi_{1b}^* = \pi - \arcsin(\Omega/\mu_1)$ $(\pi/2 \leqslant \phi_{1b}^* \leqslant \pi)$. These points exist only if $0 \leqslant \Omega/\mu_1 \leqslant 1$. When $\Omega/\mu_1 > 1$, there is not synchronism.

The local stability of $(\phi_{1a}^*, 0, 0)$ and $(\phi_{1b}^*, 0, 0)$ is determined from the eigenvalues $\lambda_{1,2,3}$ of the Jacobian matrix related to the system (7) linearized around each point. Hartman–Grobman Theorem states that an equilibrium point is locally asymptotically stable when all eigenvalues have negative real parts (e.g. [21]). For the system (7), the eigenvalues $\lambda_{1,2,3}$ are the roots of the characteristic equation:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \tag{8}$$

where $a_1 = k_1$, $a_2 = 1 + \mu_1 \cos \phi_1^*$ and $a_3 = \mu_1 \cos \phi_1^*$. According to Routh–Hurwitz Criterion (e.g. [22]), all eigenvalues have negative real parts if $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1 a_2 > a_3$. Here, this last condition corresponds to:

$$k_1 > k_{c1} \equiv \frac{\mu_1 \cos \phi_1^*}{1 + \mu_1 \cos \phi_1^*} \tag{9}$$

Therefore, for $k_1 = 0$, both equilibrium points are unstable and there is a subcritical saddle–saddle bifurcation