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Fast communication

Comments on "A linear prediction method for parameter estimation of damped sinusoids"

Qi Cheng

School of Engineering, University of Western Sydney, Locked Bag 1797, Penrith MC NSW 1797, Australia

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ABSTRACT

For the parameter estimation of damped sinusoids, the linear prediction (LP) was recently applied to a forward–backward expanded-data matrix by Kannan and Kundu [Estimating parameters in the damped exponential model, Signal Processing 81 (2001) 2343–2351]. This version of LP will be called the expanded linear prediction (ELP). A key claim of that paper is that ELP offers a comparable performance as LP. The purpose of this letter is to prove that this claim does not hold true particularly for weakly damped sinusoids.

1. Introduction

The observed data in noise with underlying damped sinusoids can be represented as

$$\hat{x}(n) = \sum_{i=1}^{I} r_i z_i^n + w(n) \stackrel{\text{def}}{=} x(n) + w(n), \quad n = 1, \ldots, N,$$
$$z_i = e^{-\alpha_i + j\omega_i} \tag{1}$$

where r_i ($|r_i|>0$), $\alpha_i(>0)$ and ω_i are the (*unknown*) complex amplitude, damping factor, and frequency of the i-th sinusoid, respectively, I is the (*known*) number of sinusoids, and $w(n)$ denotes complex i.i.d. noise samples. Each noise sample is made up of independent real and imaginary parts of Gaussian distribution with mean zero and variance $\sigma^2/2$, $z_1, \ldots, z_I$ are distinct. In (1), $\hat{x}$ denotes a noisy measurement of a quantity x. In this letter, the same notation is also used to represent an estimate of a parameter.

A popular problem in the last decade was to estimate damping factors and frequencies $\alpha_i, \omega_i, i = 1, \ldots, I$ from the collected noisy data $\{\hat{x}(n)\}_{n=1}^{N}$. For this problem, the matrix pencil (MP) [1] and (the original) linear prediction (LP) [2] can only be applied to the following data matrix (an $(N-L) \times L$ Hankel matrix)

$$\mathbf{X} = \begin{bmatrix} x(2) & \ldots & x(L+1) \\ x(3) & \ldots & x(L+2) \\ \vdots & \ddots & \vdots \\ x(N-L+1) & \ldots & x(N) \end{bmatrix} \tag{2}$$

The authors of [3], applied the concept of LP to the following forward–backward[1] (FB) expanded-data matrix

$$\dot{\mathbf{X}} = [\mathbf{X}, \mathbf{J}_{N-L}\mathbf{X}^*\mathbf{J}_L] \tag{3}$$

where $\mathbf{J}_k$ is an exchange (anti-diagonal) matrix of dimensions $k \times k$ and the superscript $*$ denotes conjugation. The resulting method is expanded linear prediction (ELP) and a summary is given in Appendix B (refer to [3] for more details). ELP requires that $\dot{\mathbf{X}}$ should have a rank equal to $2I$ according to (12) in Appendix B (also see [3, (15)]). In $\dot{\mathbf{X}}$, ELP employs a "backward" signal in addition to the LP's forward signal. The intention is to provide more data for

[1] One should note that the performance enhancing FB averaging for non-damped sinusoids was no longer applicable and not used by ELP.

E-mail address: q.cheng@uws.edu.au

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Table 1
Experimental/theoretical variances and means of the MP and LP estimates

	MP, $L=18$			LP, $L=17$		
	Exp	Theo	Mean	Exp	Theo	Mean
SNR = 10 dB						
α_1	3.429E −5	3.097E −5	0.0220	3.9374E −5	3.69E −5	0.019114
ω_1	3.526E −5	3.097E −5	2.0007	3.6207E −5	3.69E −5	2.0005
α_2	2.641E −5	2.361E −5	0.0118	3.0408E −5	2.97E −5	0.0095747
ω_2	2.295E −5	2.361E −5	1.4995	2.9922E −5	2.97E −5	1.4997
SNR = 15 dB						
α_1	0.976E −5	0.9795E −5	0.0206	1.2406E −5	1.17E −05	0.019608
ω_1	1.076E −5	0.9795E −5	2.0003	1.1399E −5	1.17E −05	2.0002
α_2	0.749E −5	0.7467E −5	0.0106	9.6057E −6	9.40E −06	0.0098474
ω_2	0.711E −5	0.7467E −5	1.4998	9.4509E −6	9.40E −06	1.4999
SNR = 20 dB						
α_1	3.000E −6	3.097E −6	0.0201	3.9173E −6	3.69E −06	0.019814
ω_1	3.360E −6	3.097E −6	2.0001	3.5996E −6	3.69E −06	2.0001
α_2	2.292E −6	2.361E −6	0.0102	3.0373E −6	2.97E −06	0.0099415
ω_2	2.240E −6	2.361E −6	1.4999	2.9908E −6	2.97E −06	1.5
SNR = 25 dB						
α_1	0.94352E −6	0.980E −6	0.020017	1.5808E −6	1.54E −6	0.019872
ω_1	1.058E −6	0.980E −6	2	1.603E −6	1.54E −6	2
α_2	0.715E −6	0.747E −6	0.010068	1.2763E −6	1.25E −6	0.0099602
ω_2	0.70874E −6	0.747E −6	1.5	1.24E −6	1.25E −6	1.5

$r_1 = r_2 = 2e^{j0}$, $(\alpha_1, \omega_1) = (0.02, 2.0)$ and $(\alpha_2, \omega_2) = (0.01, 1.5)$. $N = 25$.

ELP to work on than LP. However, when damping factors tend to zero, the backward signal becomes indistinguishable from the forward signal, and both are essentially sinusoids with no distinguishable direction. Hence in this case, $\dot{\mathbf{X}}$ contains twice the same data, with its rank being halved from $2I$ to I. This feature of $\dot{\mathbf{X}}$ makes the ELP problem ill-conditioned and thus more susceptible to noise, for *weakly damped* sinusoids.

In [3], ELP was compared with MP and LP. A key claim (in the last sentence on p. 2248 and the first sentence on p. 2250) is that ELP offers a similar performance as MP and LP, drawn from the simulation results in Table 2 of [3]. However, the MP and LP estimates in that table are evidently biased for damping factors even at high SNRs (e.g., 25 dB) and therefore their variances can not be taken as a measure of accuracy. We rerun simulations in Table 2 of [3] for MP and LP. The signal parameters are $r_1 = r_2 = 2e^{j0}$, $(\alpha_1, \omega_1) = (0.02, 2.0)$ and $(\alpha_2, \omega_2) = (0.01, 1.5)$. Table 1 lists our experimental variances along with theoretical ones based on expression (4.15) in [1] and (37) in [4] where $\mathrm{SNR} \stackrel{\mathrm{def}}{=} -10\log\sigma^2$. Experimental variance was calculated from the formula $\sum_{j=1}^{N_{\mathrm{sim}}}(\Delta\eta_i^j)^2/N_{\mathrm{sim}}$, where $\Delta\eta_i^j = \hat{\eta}_i^j - \eta_i$ denotes the perturbation on an estimate $\hat{\eta}_i^j$ of a parameter η_i given by MP or LP in the j-th simulation trial, η_i denotes either α_i or ω_i, and $N_{\mathrm{sim}} = 500$ is the total number of trials for calculating one experimental variance. In each trial, $N = 25$ data samples were collected and MP and LP yielded their estimates from the same set of noisy data. Theoretical variance was defined as $var(\Delta\eta_i) \stackrel{\mathrm{def}}{=} E\{(\Delta\eta_i)^2\}$ where $\Delta\eta_i$ is an estimation perturbation expression involving noise samples and other quantities depending on signal parameters and E stands for statistical expectation.

Our estimates are nearly unbiased at the medium to high SNRs. More importantly, unlike the results in Table 2 of [3], there is a close agreement between our experiment and theory. Comparing our Table 1 with Table 2 of [3], it is clear that the simulations in the latter were not conducted properly. The choice of the window length L in [3] is also problematic. For example, for $L = 20$ in [3], the condition number[2] of $\dot{\mathbf{X}}$ is the highest, leading to biased ELP estimates (particularly at SNR = 20 dB, 25 dB).

This letter intends to present a correct evaluation of ELP against MP and LP. In all our simulations, MP was found to perform slightly better than LP under the same value of L, and therefore, it will not be included in the forthcoming comparison for the sake of brevity.

We compared ELP with LP via simulations and evaluation of theoretical variances, for varying SNRs as well as varying damping factors. SNR is defined in the same way as in Table 1. The expressions of the theoretical ELP variances had been derived, in parallel to (19)–(24) in [4], and are presented in Appendix C. We considered two ($I = 2$) equal-power sinusoids. N and N_{sim} took the same values as that in Table 1.

Search for an optimal window length L was conducted for two weakly damped sinusoids with the same signal parameters as in Table 1 at SNR = 20 dB. Results are

[2] The condition number is defined as the ratio of the largest to the smallest positive singular values of a matrix.