Signal Processing 92 (2012) 288–293

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

Fast communication

# Signal decomposition by the S-method with general window functions

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ARTICLE INFO

*Article history:*
Received 2 April 2011
Received in revised form
31 May 2011
Accepted 20 June 2011
Available online 29 June 2011

*Keywords:*
The S-method
Time–frequency representation
Signal decomposition
Window function

ABSTRACT

In this letter, we propose a time–frequency signal decomposition algorithm based on the S-method. The proposed algorithm is an extension of the algorithm in [1], so that a general case of the window function used in calculating the S-method can be dealt with. The effect of window function is explicitly verified, based on which a penalized squared error minimization is used to recover the signal matrix. The signal components are reconstructed by eigen decomposition on the recovered signal matrix. Simulation validates the effectiveness of the proposed algorithm.



## 1. Introduction

Time–frequency representation (TFR) provides a key tool for nonstationary signal processing. Short time Fourier transform (STFT) and Wigner distribution (WD) are the most important classical TFRs with wide applications [2]. However, their limitations are also well known, e.g., the limited time–frequency resolution of STFT and the interference problem of WD. In particular, to reduce WD's interference, a number of extended TFRs were proposed, such as the polynomial time–frequency distributions [3] and the generalized time–frequency distributions [4]. The S-method proposed by Stanković [5] is a TFR that is closely related to STFT and WD, and it reduces both limitations of these two in certain extent. Several attractive properties of the S-method are listed as below:

1. it can be calculated based on the short time Fourier transform (STFT),
2. it approaches the Wigner distribution (WD) when the window function used in STFT is a rectangular window of the signal's length, and
3. it is a "linear" TFR in the sense that the S-method of multi-component signal equals to the sum of each component's S-method, given that the components do not overlap on the spectrogram.

Thanks to above properties, the S-method has been applied to many signal processing problems, e.g., SAR/ISAR imaging [6] and the analysis of HF radar signal in sea-clutter [1].

In this paper, we focus on time–frequency signal decomposition by the S-method. Such study has been investigated in [1]. The main idea of the signal decomposition algorithm is that the S-method of a signal equals to the sum of the WDs of its components. Thus, formally, Stanković et al. [1] only address the case where the STFT is performed with a rectangular window of the signal's length, but leaves a problem: how to recover the components of a signal when a general (short) window function is used.

Since the S-method is calculated from STFT, on which the resolution of spectrogram affects the S-method and the signal decomposition based on. In this sense, the rectangular used in [1] is not preferred, because its window generally leads to low time–frequency resolutions. In particular, for multi-component signals, the use of rectangular window can make them inseparable on

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0165-1684/$ - see front matter 
doi:10.1016/j.sigpro.2011.06.013

spectrogram. To address this problem, we propose an extended signal decomposition algorithm based on the S-method, which can deal with general (short) window functions. We prove that the window function has a "masking" effect on the signal matrix, and propose a penalized squared error minimization to recover the signal matrix. The reconstruction of signal components is done by eigen decomposition on the recovered signal matrix. In Section 2, after a brief review of the S-method, we present the extended S-method based signal decomposition algorithm. Section 3 reports simulation results. And Section 4 concludes this letter.

## 2. Signal decomposition with general window functions

### 2.1. A brief review of the S-method

Given the STFT of signal $x(n)$, $n=0,\dots,N$,

$$\mathrm{STFT}(n,k)=\sum_{m=-M/2}^{M/2} w(m)x(n+m)e^{-j(2\pi/(M+1))mk}, \tag{1}$$

where $w(m)$, $m=-M/2,\dots,M/2$, is the window function, the S-method of $x(n)$ is defined by [5]

$$\mathrm{SM}(n,k)=\frac{1}{N+1}\sum_{l=-L}^{L}\mathrm{STFT}(n,k+l)\mathrm{STFT}^*(n,k-l), \tag{2}$$

where $L$ is a parameter determined by the time–frequency bandwidth of signal $x(n)$, and $\mathrm{STFT}^*(n,k)$ is the complex conjugate of $\mathrm{STFT}(n,k)$.

When $x(n)$ has multiple components, i.e., $x(n)=\sum_{i=1}^{c} x_i(n)$ and the components do not overlap on the spectrogram, the parameter $L$ in (2) can be properly chosen [1] such that

$$\mathrm{SM}(n,k)=\sum_{i=1}^{c}\mathrm{SM}_i(n,k), \tag{3}$$

where $\mathrm{SM}_i(n,k)$ is the S-method of $x_i(n)$.

### 2.2. The effect of window function

Denote signal $x(n)$ by a column vector $\mathbf{x}$, and let $\mathbf{X}=\mathbf{x}\mathbf{x}^*$, which we call the signal matrix. It is shown that, when $w(m)$ in (1) is chosen as a rectangular window of the signal's length, i.e., $w(m)=1$, $m=-N/2,\dots,N/2$, the following matrix $\mathbf{R}$ [1]:

$$\mathbf{R}(n_1,n_2)=\frac{1}{N+1}\sum_{m=-N/2}^{N/2}\mathrm{SM}\left(\frac{n_1+n_2}{2},m\right)e^{j(2\pi/(N+1))m(n_1-n_2)} \tag{4}$$

equals the signal matrix $\mathbf{X}$, i.e.,

$$\mathbf{R}=\mathbf{x}\mathbf{x}^*. \tag{5}$$

However, for a general choice of the window function $w(m)$, above equation does not hold.

To verify the effect of $w(m)$ on (5), we introduce a notation $\mathbf{W}$, called the window matrix,

$$\mathbf{W}(n_1,n_2)=\begin{cases} w^2((n_1-n_2)/2), & |(n_1-n_2)/2|\le M/2 \\ 0 & \text{else}. \end{cases} \tag{6}$$

The following theorem shows explicitly the effect of $\mathbf{W}$ on the relationship between the matrix $\mathbf{R}$ and the signal matrix $\mathbf{X}$ (see Appendix for the proof).

**Theorem 1.** *Suppose that* $\mathrm{SM}_w(n,k)$ *is the S-method of* $x(n)$, $n=0,1,\dots,N$, *computed with window* $w(m)$, $m=-M/2,\dots,M/2$, *then the matrix* $\mathbf{R}$ *in* (4) *and the window matrix* $\mathbf{W}$ *in* (6) *satisfy*

$$\mathbf{R}=\mathbf{W}\odot\mathbf{X} \tag{7}$$

*where* $\odot$ *is entry-wise multiplication.*

As a special case of Theorem 1, when $w(m)$ is a rectangular window of the signal's length, $\mathbf{W}$ has all entries 1. And thus (7) degenerates to (5). Otherwise, however, $w(m)$ introduces a "mask", $\mathbf{W}$, on the signal matrix $\mathbf{X}$.

Furthermore, for multi-component signal $x(n)=\sum_{i=1}^{c} x_i(n)$, we define the signal matrix as

$$\mathbf{X}=\sum_{i=1}^{c}\mathbf{x}_i\mathbf{x}_i^*, \tag{8}$$

and Theorem 1 still holds. Actually, by the "linear" property (3) of the S-method, and the definition (4) of the matrix $\mathbf{R}$, it can be verified that

$$\mathbf{R}=\sum_{i=1}^{c}\mathbf{R}_i=\sum_{i=1}^{c}\mathbf{W}\odot\mathbf{x}_i\mathbf{x}_i^*=\mathbf{W}\odot\sum_{i=1}^{c}\mathbf{x}_i\mathbf{x}_i^*, \tag{9}$$

where $\mathbf{R}_i$ corresponds to the $\mathrm{SM}_i(n,k)$ of $x_i(n)$ and we have applied Theorem 1 to each pair of $\mathbf{R}_i$ and $x_i(n)$.

### 2.3. Recovery of signal matrix **X**

To reconstruct the components of signal $x(n)$, first we need recover the signal matrix $\mathbf{X}$ from (7). A straightforward way for this recovery is to use the following feasible problem:

$$\begin{aligned} &\text{find} \quad \mathbf{X} \\ &\quad \text{s.t.} \quad \mathbf{W}\odot\mathbf{X}=\mathbf{R}, \quad \mathbf{X}\in\mathbf{H}^{(N+1)\times(N+1)}, \end{aligned} \tag{10}$$

where $\mathbf{H}^{(N+1)\times(N+1)}$ denotes the set of all Hermitian matrices of size $(N+1)\times(N+1)$. Note that $\mathbf{X}=\sum_{i=1}^{k}\mathbf{x}_i\mathbf{x}_i^*$ implies $\mathbf{X}$ is a Hermitian matrix. We further reformulate (10) as a minimization of squared error

$$\begin{aligned} &\min \quad \|\mathbf{W}\odot\mathbf{X}-\mathbf{R}\|_F^2 \\ &\quad \text{s.t.} \quad \mathbf{X}\in\mathbf{H}^{(N+1)\times(N+1)}. \end{aligned} \tag{11}$$

**Algorithm 1** (*Recovery of Signal Matrix* $\boldsymbol{X}$).

Initialization $X_0$
**for** m=1,2,... **do**
 Calculate $J_m=\|\mathbf{W}\odot\mathbf{X}_m-\mathbf{R}\|_F^2+\log\det(\mathbf{X}_m+\delta I)$.
 Calculate $\nabla(\mathbf{X}_m)=2\mathbf{W}\odot(\mathbf{W}\odot\mathbf{X}_m-\mathbf{R})+\rho(\mathbf{X}_m+\delta\mathbf{I})^{-H}$.
 Update $\mathbf{X}_{m+1}=\mathbf{X}_{m+1}-\eta\nabla(\mathbf{X}_m)$.
 Decompose $\mathbf{X}_{m+1}=\sum_{i=1}^{N+1}\lambda_i\xi_i\xi_i^*$, and let $\mathbf{X}_{m+1}=\sum_{\lambda_i>=0}\lambda_i\xi_i\xi_i^*$.
 Stop if $|J(\mathbf{X}_{m+1})-J(\mathbf{X}_m)|/|J(\mathbf{X}_m)|<\varepsilon$.
**end for**

where $\|\cdot\|_F$ is the Frobenius norm of a matrix. A problem with (11) is that its solution is generally not unique.