Signal Processing 91 (2011) 2259–2267

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

On the asymptotic distribution of GLR for impropriety of complex signals

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ARTICLE INFO

Article history:
Received 23 September 2010
Received in revised form
1 April 2011
Accepted 3 April 2011
Available online 17 April 2011

Keywords:
Generalized likelihood ratio (GLR)
Receiver operating characteristics (ROC)
Asymptotic distribution of circularity coefficients estimate
Improper
Second-order noncircular complex random variables

ABSTRACT

In this paper, the problem of testing impropriety (i.e., second-order noncircularity) of a sequence of complex-valued random variables (RVs) based on the generalized likelihood ratio test (GLRT) for Gaussian distributions is considered. Asymptotic (w.r.t. the data length) distributions of the GLR are given under the hypothesis that RVs are proper or improper, and under the true, not necessarily Gaussian distribution of the RVs. The considered RVs are independent but not necessarily identically distributed: assumption which has never been considered until now. This enables us to deal with the practical important situations of noncircular RVs disturbed by residual frequency offsets and additive circular noise. The receiver operating characteristic (ROC) of this test is derived as byproduct, an issue previously overlooked. Finally illustrative examples are presented in order to strengthen the obtained theoretical results.

1. Introduction

For complex-valued RVs, many papers (see, e.g., [1–4]) show that significant performance gains can be achieved by second-order algorithms based on both $\mathbf{C}_x=E(\mathbf{x}\mathbf{x}^T)$ and $\mathbf{R}_x=E(\mathbf{x}\mathbf{x}^H)$. They exploit the statistical information contained in $\mathbf{C}_x$, provided it is nonzero in addition to that contained in the standard covariance matrix $\mathbf{R}_x$. These algorithms face an additional complexity. Moreover, some such algorithms (see e.g., [5]) adapted for improper or second-order noncircular signals, i.e., with nonzero matrices $\mathbf{C}_x$, fail or suffer of too slow convergence when they are used for proper or second-order circular signals. It is thus important to adapt the processing to the properness of the observation.

Hence, the question arises as to how we can classify a signal as proper or improper. This problem is a binary hypothesis test H_0: $\mathbf{C}_x=\mathbf{0}$ versus H_1: $\mathbf{C}_x\neq\mathbf{0}$. In practice, as the parameters $\mathbf{R}_x$ and $\mathbf{C}_x$ are clearly unknown, only the GLR detector can be used. This detector was introduced independently by Ollila and Koivunen [6] and Schreier et al. [7] under the traditional assumption of independent and identically distributed Gaussian samples $(\mathbf{x}_k)_{k=1,\ldots,K}$. But in these works, its performance was illustrated by a Monte Carlo simulation only. Walden and Rubin-Delanchy [8] derived recently this GLRT as well by formulating this testing problem in terms of real-valued Gaussian random vectors. Note that they have also presented a theoretical analysis of the null asymptotic distribution of the GLR with several numerical studies based on Monte Carlo simulations for the alternative distribution under the Gaussian distribution of the signals. Furthermore, there have been recent extensions of this GLRT to

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doi:10.1016/j.sigpro.2011.04.007

non-Gaussian RVs. Authors in [9] have extended this GLRT to complex elliptically symmetric distributions, with a slight adjustment by dividing it with an estimated scaled standardized fourth-order moment. Then in [10], a GLRT based on complex generalized Gaussian distributions have been provided. These extensions make the GLRT more robust to non-Gaussian distributions, but surprisingly they do not improve the performance for sub-Gaussian distributions [10], which include the majority of applications in communications and radar.

The aim of this paper is to complement the theoretical asymptotical analysis of [8,9]. The originality of our approach consists in considering the null and alternative asymptotic distribution of the GLR derived under the Gaussian distribution, but used in practice under independent not necessarily identically Gaussian distributed data. This paper is organized as follows. The GLRT is recalled for the convenience of the reader in Section 2. The asymptotic distribution of the GLR under the hypothesis that RVs are proper or improper is considered in Section 3, using the asymptotic distributions of the circularity coefficients given in [11]. This asymptotic distribution is given in the scalar case and then extended to the multidimensional case under the assumption of independent identically not necessarily Gaussian distributed RVs. An interpretable closed-form expression of the ROC is given in the scalar case due to the simplicity of the asymptotic distribution of the GLR. Then, extension of this study to independent nonidentically distributed RVs is considered in Section 4. This enables us to deal with practical situations of noncircular RVs disturbed by residual frequency offsets and additive circular noise. Finally some illustrative examples are presented in Section 5. Note that some results of this paper have been given in [12].

The following notations are used throughout the paper. Matrices and vectors are represented by bold upper case and bold lower case characters, respectively. Vectors are by default in column orientation, while T, H and $*$ stand for transpose, conjugate transpose, conjugate, respectively. $\mathrm{vec}(\cdot)$ is the "vectorization" operator that turns a matrix into a vector by stacking the columns of the matrix one below another which is used in conjunction with the Kronecker product $\mathbf{A}\otimes\mathbf{B}$ as the block matrix whose (i, j) block element is $a_{i,j}\mathbf{B}$ and with the vec-permutation matrix $\mathbf{K}$ which transforms $\mathrm{vec}(\mathbf{C})$ to $\mathrm{vec}(\mathbf{C}^T)$ for any matrix $\mathbf{C}$.

2. Generalized likelihood ratio decision rule

We assume that $(\mathbf{x}_k)_{k=1,\ldots,K}\in\mathbb{C}^N$ is a realization of K independent identically zero-mean complex Gaussian distributed RVs. Their covariance matrices $\mathbf{R}_x=E(\mathbf{x}\mathbf{x}^H)$ and $\mathbf{C}_x=E(\mathbf{x}\mathbf{x}^T)$ are unknown. Consider the following binary composite hypothesis testing problem:

$$H_0:\mathbf{C}_x=\mathbf{0},\quad \mathbf{R}_x,$$

$$H_1:\mathbf{C}_x\neq\mathbf{0},\quad \mathbf{R}_x.$$

In the likelihood ratio, the GLR replaces the unknown parameters $\mathbf{R}_x$ and $\mathbf{C}_x$ by their maximum likelihood (ML) estimates. It is thus straightforward to derive its expression which is given by [6,7]

$$L(\mathbf{x},K)\stackrel{\mathrm{def}}{=}\frac{p((\mathbf{x}_k)_{k=1,\ldots,K};\hat{\mathbf{R}}_x,\hat{\mathbf{C}}_x,H_1)}{p((\mathbf{x}_k)_{k=1,\ldots,K};\hat{\mathbf{R}}_x,\mathbf{0},H_0)}=\frac{\det(\hat{\mathbf{R}}_x)^K}{\det(\hat{\mathbf{R}}_{\tilde{x}})^{K/2}}\tag{1}$$

with $\hat{\mathbf{R}}_x\stackrel{\mathrm{def}}{=}(1/K)\sum_{k=1}^K\mathbf{x}_k\mathbf{x}_k^H$ and $\hat{\mathbf{R}}_{\tilde{x}}\stackrel{\mathrm{def}}{=}(1/K)\sum_{k=1}^K\tilde{\mathbf{x}}_k\tilde{\mathbf{x}}_k^H$ where $\tilde{\mathbf{x}}_k\stackrel{\mathrm{def}}{=}[\mathbf{x}_k^T,\mathbf{x}_k^H]^T$. The GLRT decides H_1 if

$$L(\mathbf{x},K)>\lambda\tag{2}$$

and otherwise H_0. In the scalar case $N=1$, the GLRT is the UMP linearly invariant test [8]. But note that no uniformly most powerful (UMP) $\mathbb{C}$ linearly[2] invariant test for impropriety exists for $N>1$ [8]. It becomes especially simple

$$L(\mathbf{x},K)=(1-\hat{\gamma}_x^2)^{-K/2}\tag{3}$$

with $\hat{\gamma}_x=|(1/K)\sum_{k=1}^K x_k^2|/(1/K)\sum_{k=1}^K|x_k|^2$ is the ML estimate [13,11] of the circularity coefficient $\gamma_x\stackrel{\mathrm{def}}{=}|E(x_k^2)|/E|x_k|^2$. By the increasing monotony of (3), the GLRT decides H_1 if

$$\hat{\gamma}_x>\lambda',\tag{4}$$

which is quite intuitive.

3. Asymptotic distribution of GLR for IID observations

Throughout this section, this GLRT is used for independent identically zero-mean nonnecessarily Gaussian distributed RVs $(\mathbf{x}_k)_{k=1,\ldots,K}$. For such non-Gaussian RVs, decision rule (2) is no longer a GLRT. However, it generally provides good performance in practice (see e.g., for the detection of a known signal corrupted by noncircular interference [14]) and is simple to implement.

3.1. Scalar complex random variable

Let x_k be a scalar valued RV of arbitrary distribution with finite fourth-order moments. We suppose that under H_0, x_k is circular up to the fourth-order.[3] Then, the following result is proved in the Appendix:

Result 1. Under the respective hypothesis H_0 and H_1, the following convergences in distribution hold when $K\to\infty$

$$\sqrt{\frac{K}{1+\frac{\kappa_x}{2}}}\hat{\gamma}_x\xrightarrow{\mathcal{L}}\mathcal{R}(1),\tag{5}$$

$$\sqrt{K}(\hat{\gamma}_x-\gamma_x)\xrightarrow{\mathcal{L}}\mathcal{N}(0,\sigma_\gamma^2)\quad\text{if }\gamma_x\neq 1.\tag{6}$$

In (5) and (6), $\mathcal{R}(1)$ and $\mathcal{N}(0,\sigma_\gamma^2)$ denote the Rayleigh distribution with unit scale (i.e., the chi distribution with two degrees of freedom χ_2) and the zero-mean Gaussian

[2] $\mathbb{C}$ linear transformations include rotation and scaling, but not widely linear operations.

[3] This means that not only $E(x_k^2)=0$, but also the fourth-order cumulants satisfy $\mathrm{cum}(x_k,x_k,x_k,x_k)=0$ and $\mathrm{cum}(x_k,x_k,x_k,x_k^*)=0$ [15]. We note, it is possible that $E(x_k^2)=0$ with $\mathrm{cum}(x_k,x_k,x_k,x_k)\neq 0$ or $\mathrm{cum}(x_k,x_k,x_k,x_k^*)\neq 0$. In this case, the asymptotic distribution of $\hat{\gamma}_x$ is much more involved (see the proof of Result 1 in the Appendix).