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Fast communication

Reduced biquaternion canonical transform, convolution and correlation

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ABSTRACT

The reduced biquaternion canonical transform (RBCT) is defined in this paper, which is the generalization of reduced biquaternion Fourier transform (RBFT). The Parseval's theorem related to RBCT is investigated. The concepts of reduced biquaternion canonical convolution (RBCCV) and reduced biquaternion canonical correlation (RBCCR) are defined, then the convolution and correlation theorem of RBCT are developed in this paper. All these theorems can also be seen as the generalizations of the corresponding theorem related to RBFT. Finally, the discrete form and fast algorithm of RBCT are presented, and the computation complexity is similar to FFT.

1. Introduction

Recently, many signal processing tools using the quaternions have been proposed, including quaternion Fourier transform (QFT) [1–4], quaternion wavelet transform (QWT) [5–7] and fractional quaternion Fourier transform (FrQFT) [8]. However, due to the noncommutative property of the quaternion multiplication, some important theorems cannot be generalized to quaternion signal processing tools. For example, the Parseval's theorem does not hold for FrQFT; the convolution of two quaternion signal $f(x,y)$ and $g(x,y)$ cannot be calculated by the product of their QFT [3]. In addition, the computation of QFT and FrQFT are somewhat complex. In order to overcome these drawbacks, we combine the reduced biquaterion algebra and the linear canonical transform (LCT) and propose the reduced biquaterion canonical transform (RBCT).

Linear canonical transform (LCT) [10–14] is an important tool in signal processing. Many transforms such as Fourier transform, fractional Fourier transform and the Fresnel transform are special cases of the LCT. During the last decade, there were many achievements associated with the LCT [15–21]. However, the LCT deals with real scalar signal or complex signal (analytical signal [19]), none of them processes reduced biquaternion signals. As the generalization of complex signal, reduced biquaternion signal, has one real part and three imaginary parts. Based on reduced biquaternion (RB) algebra system, we define the reduced biquaternion canonical transform, which can process reduced biquaternion signals.

In Section 2, we first give a brief introduction about quaternion and reduced biquaternion. The reduced biquaternion canonical transform is defined, which can process not only real scalar or complex signal but also reduced biquaternion signal. The Parseval's equality of RBCT is derived in Section 3. Moreover, in Sections 4 and 5, we generalize the definitions of convolution and correlation, propose the reduced biquaternion canonical convolution (RBCCV) and reduced biquaternion canonical correlation

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(RBCCR). The convolution and correlation theorems of RBCT are derived. The theorems show that the RBCCV or RBCCR can be calculated in the RBCT domain. Section 6 developed the discrete form and fast algorithm of RBCT, simulations are also implemented. Finally, conclusions are made in Section 7.

2. Preliminaries

2.1. The quaternion signal and reduced biquaternion signal

The quaternion, which is a type of hypercomplex number, was formally introduced by Hamilton in 1843. It is a generalization of complex number. We know that a complex number has two components: the real part and imaginary part. However, the quaternion has four parts, i.e., one real part and three imaginary parts. For a quaternion q, which can be written in a rectangular form as follows: $q=q_r+q_i i+q_j j+q_k k$, where $q_r,q_i,q_j,q_k \in \mathbb{R}$ and i,j,k are complex operators obeying the following rules:

$$i^2=j^2=k^2=-1, \quad ij=-ji=k, \; jk=-kj=i, \; ki=-ik=j \tag{1}$$

Correspondingly, the complex signal can be generalized to quaternion signal, i.e., $f(x,y)=f_r(x,y)+if_i(x,y)+jf_j(x,y)+kf_k(x,y)$, where $f_r(x,y),f_i(x,y),f_j(x,y),f_k(x,y) \in \mathbb{R}^2$. Some signal processing tools can process quaternion signals, including quaternion Fourier transform, quaternion wavelet transform, fractional quaternion Fourier transform and quaternion Fourier–Mellin moment [9].

However, the multiplication rule of quaternions is not commutative. T.A.Ell defined the double-complex algebra, which is similar to quaternions but with commutative multiplication [1]. In Ref. [22], reduced biquaternions (RBs) was proposed, and commutative multiplication was defined for it.

The description of RBs is $q=q_r+q_i i+q_j j+q_k k$, where $q_r,q_i,q_j,q_k \in \mathbb{R}$, the imaginary operators obey the following multiplicative rules:

$$i^2=k^2=-1, \quad j^2=1, \; ij=ji=k, \; jk=kj=i, \; ki=ik=-j \tag{2}$$

RBs can also be represented as follows: $q=q_1+q_2 j$, where $q_1=q_r+q_i i$, $q_2=q_j+q_k i$.

Another representation of RBs is e_1-e_2 form [23], i.e., $q=q_1+q_2 j \equiv q_{1+2}e_1+q_{1-2}e_2$, where $q_{1+2}=q_1+q_2$, $q_{1-2}=q_1-q_2$, $e_1=(1+j)/2$, $e_2=(1-j)/2$. We will use e_1-e_2 form of RB signals to develop the fast algorithm of RBCT in Section 6.

In Ref. [24], the norm and conjugate of RBs were defined. The norm of a reduced biquaternion $q=q_r+q_i i+q_j j+q_k k$ is

$$\|q\|=[(q_r^2+q_i^2+q_j^2+q_k^2)^2-4(q_r q_j+q_i q_k)^2]^{\frac{1}{4}} \tag{3}$$

the conjugate of q is

$$\overline{q}=\|q\|^2 q^{-1}=\|q\|^2/q \tag{4}$$

The two-dimensional reduced biquaternion signal $f(x,y)$ is defined as follows: $f(x,y)=f_r(x,y)+if_i(x,y)+jf_j(x,y)+kf_k(x,y)$. For any two reduced biquaternion signals $f(x,y)$ and $g(x,y)$, $f(x,y)=f_1(x,y)+f_2(x,y)j$, $g(x,y)=g_1(x,y)+g_2(x,y)j$, where $f_1(x,y)=f_r(x,y)+if_i(x,y)$, $f_2(x,y)=f_j(x,y)+if_k(x,y)$, $g_1(x,y)=g_r(x,y)+ig_i(x,y)$, $g_2(x,y)=g_j(x,y)+ig_k(x,y)$. According to the multiplication rule of reduced biquaternion, we have

$$f(x,y)g(x,y)=g(x,y)f(x,y)=[f_1(x,y)g_1(x,y)+f_2(x,y)g_2(x,y)] + j[f_1(x,y)g_2(x,y)+f_2(x,y)g_1(x,y)] \tag{5}$$

If $f(x,y)$ and $g(x,y)$ are quaternion signals, Eq. (5) will not holds, i.e., $f(x,y)g(x,y) \neq g(x,y)f(x,y)$.

2.2. Definition of the RBCT

Definition 1. Let $f(x,y)$ be a reduced biquaternion signal with condition $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\|f(x,y)\|^2\,dx\,dy<\infty$, then the RBCT of $f(x,y)$ with parameters A_1 and A_2 is defined as

$$F_{i,k}^{A_1,A_2}(u,v) \triangleq L_{i,k}^{A_1,A_2}[f(x,y)](u,v) = \begin{cases} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)K_i^{A_1}(x,u)K_k^{A_2}(y,v)\,dx\,dy, & b_1 b_2 \neq 0 \\ \sqrt{d_1 d_2}f(d_1 u,d_2 v)e^{i(c_1 d_1/2)u^2}e^{k(c_2 d_2/2)v^2}, & b_1 b_2=0 \end{cases} \tag{6}$$

where

$$K_i^{A_1}(x,u)=\sqrt{\frac{1}{i2\pi b_1}}e^{i((d_1/2b_1)u^2+(a_1/2b_1)x^2-(1/b_1)ux)} \tag{7}$$

$$K_k^{A_2}(y,v)=\sqrt{\frac{1}{k2\pi b_2}}e^{k((d_2/2b_2)v^2+(a_2/2b_2)y^2-(1/b_2)vy)} \tag{8}$$

$A_1=(a_1,b_1,c_1,d_1)$, $A_2=(a_2,b_2,c_2,d_2)$, $a_s,b_s,c_s,d_s \in \mathbb{R}, s=1,2$. $a_1 d_1-b_1 c_1=1$, $a_2 d_2-b_2 c_2=1$.

If $A_1=A_2=(0,1,-1,0)$, the RBCT will be reduced to RBFT [24,25]. If $A_1=(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)$, $A_2=(\cos\beta,\sin\beta,-\sin\beta,\cos\beta)$, we can define the fractional reduced biquaternion Fourier transform (FrRBFT). In Ref. [8], the fractional quaternion Fourier transform was defined. Because the definition of FrQFT is based on quaternion algebra, so the commutative property of multiplication does not hold. This drawback leads to that the FrQFT does not satisfy parseval's principle any longer and the computation of the FrQFT is more complex. As a special case of RBCT, the definition of FrRBFT is based on commutative quaternion algebra. So the FrRBFT can overcome the drawback mentioned above. If $f(x,y)$ is the real scalar or complex signal, and set the imaginary operator $k=i$ in Eq. (6), then the RBCT will be reduced to traditional two-dimensional LCT. So the RBCT can also be seen as the generalization of LCT [19,26,27].

The inverse transform of the RBCT is also given by a reduced biquaternion canonical transform with parameters: $A_1^{-1}=(d_1,-b_1,-c_1,a_1)$ and $A_2^{-1}=(d_2,-b_2,-c_2,a_2)$, that is

$$f(x,y)=\begin{cases} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{i,k}^{A_1,A_2}(u,v)K_i^{A_1^{-1}}(u,s)K_k^{A_2^{-1}}(v,t)\,du\,dv, & b_1 b_2 \neq 0 \\ \sqrt{a_1 a_2}f(a_1 x,a_2 y)e^{-i(a_1 c_1/2)x^2}e^{-k(a_2 c_2/2)y^2}, & b_1 b_2=0 \end{cases} \tag{9}$$