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Mean curvature flow on graphs for image and manifold restoration and enhancement

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ABSTRACT

In this paper, we propose an adaptation and a transcription of the mean curvature level set equation on the general discrete domain, a weighted graph. For this, we introduce perimeters on graphs using difference operators and define the curvature as the first variation of these perimeters. Then we propose a morphological scheme that unifies both local and nonlocal notions of mean curvature on Euclidean domains. Furthermore, this scheme allows to extend the mean curvature applications to process images, manifolds and data which can be represented by graphs.

1. Introduction

In this paper, we present an adaptation of mean curvature flow level set equation on weighted graphs using the framework of Partial difference equation [7,9]. This adaptation aims to extend the mean curvature equation applications to any discrete data that can be represented by graphs. Moreover, it leads to a finite difference equation with data depending on coefficients whose solution gives rise to a new class of morphological operators for data restoration and enhancement.

1.1. Context and motivation

With the advent of our digital world, many different kinds of data are now available (images, meshes, social networks, etc.) that do not necessarily lie on a Cartesian grid and that can be irregularly distributed. To represent these data, the most natural and flexible representation consists in using weighted graphs by modeling neighborhood relationships. Processing these data on graphs is then a major challenge for image processing and machine learning communities, to address many applications, such as denoising, enhancement and clustering.

Historically, the main tools for the study of graphs or networks come from combinatorial and graph theory. Recently, there has been increasing interest in the investigation of two major mathematical tools for signal and image analysis, which are PDEs and wavelet on graph [8]. In particular, the PDE on graph was used in different applications that include filtering, denoising, segmentation and clustering, see [7,9,12,16–20] and references therein for more details. In recent papers, the study of PDEs has appeared to be a subject of interest, dealing with the existence and qualitative behavior of the solutions [13,14]. In this work, we consider the Partial difference Equations (PdEs) method that mimics PDEs on graphs, by replacing differential operators by difference operators on graphs.

Following these works on PdEs on graphs [12], we propose to extend the notion of mean curvature to discrete settings and to show the relation between this mean

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curvature and local and nonlocal forms of curvature in Euclidean domains. We also extend mean curvature applications to any discrete data that can be represented by graphs to solve many problems in image and manifold processing.

1.2. Short overview on level set mean curvature flow

In the last few decades, there has been increasing interest in mean curvature flows with applications in image processing (denoising, enhancement, segmentation); many papers have been devoted to its numerical algorithms. These algorithms are related to finite difference methods on uniform grids, threshold dynamics [22] and mathematical morphology using Min/Max operators on game theoretical approach, see [4–6] for more details.

The level set formulation to describe the curve evolution has been introduced by Osher-Sethian [1]. It provides well-known advantages such as treating self-intersections or topological changes and can be easily extended to $\mathbb{R}^d$ with $d \geq 1$. Given a parametrized curve $\Gamma: [0,1] \to \Omega$, evolving on a domain $\Omega \subset \mathbb{R}^d$ due to the effect of a scalar field $\mathcal{F}: \Omega \to \mathbb{R}$. The level set method aims to find a function $f(x,t)$ such that at each time t the evolving curve Γ_t can be provided by the 0-level set of $f(x,t)$. In other words $\Gamma_t = \{x | f(x,t) = 0\}$ and the curve evolution can be done solving

$$\frac{\partial f}{\partial t} = \mathcal{F} |\nabla f(x,t)|,$$

with an initial condition $f(x,0) = f_0(x)$, the initial embedding of Γ. In the context of image processing, f_0 corresponds to the given noisy image or to an implicit representation of a front (surface). When the normal velocity $\mathcal{F}$ also depends on the spatial derivative of the normal vector, we obtain the following mean curvature level set equation:

$$\frac{\partial f}{\partial t} = \mathcal{K} |\nabla f(x,t)|, \tag{1}$$

where $\mathcal{K} = \mathrm{div}(\nabla f / |\nabla f|)$, the quantity $|\nabla f(x,t)|$ is the module of gradient.

1.3. Contributions

Our main contributions are as follows. We propose to define the notion of discrete weighted perimeters using a family of discrete gradients on graphs. As in the continuous setting, we introduce the notion of nonlocal curvature as the first variation of the discrete perimeters. We show that our formulation unifies both local and nonlocal notions of the curvature.

The transcription of the level set equation on graphs by replacing curvature and gradient leads to a PdE. The new numerical scheme we propose leads a morphological approach alternating dilation and erosion processes as

$$\frac{\partial f}{\partial t} = \max(k_w(u), 0) |\nabla_w^+ f(u)|_p + \min(k_w(u), 0) |\nabla_w^- f(u)|_p$$

where $\mathcal{K}_w, \nabla_w^+, \nabla_w^-$ are respectively the nonlocal curvature and upwind gradient on a given weighted graph $G = (V, E, w)$.

Finally, we show that our approach can deal with different types of applications including image filtering, images on mesh filtering and 3D surface smoothing.

Remark. The term nonlocal, applied to our discrete operators, is related to the non-locality of data defined on Euclidean domains (as images). Indeed, by graph construction, these operators can mimic non-local operators defined on the continuous domain. Then, this term is used as a reference to the continuous case [21] where it means that each element can interact with every other element in the domain (and not only adjacent ones), and should not be confused with the one in non-local filtering (that uses patches).

1.4. Paper organization

The rest of this paper is organized as follows. Section 2 presents a general definition of Partial difference Equations on weighted graph. Section 3 presents our new formalism of the Mean Curvature. Section 4 presents some experiments. Finally, Section 5 concludes this paper.

2. Partial difference equations on graphs

2.1. Notations and definitions

We begin briefly by reviewing some basic definitions and operators on weighted graphs. See [2,9] for more details.

Let us consider the general situation where any discrete domain can be viewed as a weighted graph. A weighted graph $G = (V, E, w)$ consists of a finite set V of N vertices and of a finite set $E \subseteq V \times V$ of edges. Let (u, v) be the edge that connects vertices u and v. An undirected graph is weighted if it is associated with a weight function $w: V \times V \to [0,1]$. The weight function represents a similarity measure between two vertices of the graph. According to the weight function, the set of edges is defined as $E = \{(u,v) | w(u,v) > 0\}$. The degree of a vertex u is defined as $\mu(u) = \sum_{v \sim u} w(u,v)$. The neighborhood of a vertex u (i.e., the set of vertices adjacent to u) is denoted $N(u)$. Notation $v \sim u$ means that the vertex v is adjacent to u. Let $\mathcal{H}(V)$ be the Hilbert space of real valued functions on the vertices of the graph. Each function $f: V \to \mathbb{R}$ of $\mathcal{H}(V)$ assigns a real value $f(u)$ to each vertex $u \in V$. Similarly, let $\mathcal{H}(E)$ be the Hilbert space of real valued functions defined on the edges of the graph. These two spaces are endowed with the following inner products: $\langle f, h \rangle_{\mathcal{H}(V)} = \sum_{u \in V} f(u) g(u) \mu(u)$ with $f, g \in \mathcal{H}(V)$, and $\langle F, H \rangle_{\mathcal{H}(E)} = \sum_{u \in V} \sum_{v \in V} F(u,v) G(u,v) w(u,v)$ where $F, G \in \mathcal{H}(E)$.

Given a function $f: V \to \mathbb{R}$, the $\mathcal{L}_p$ norm of f is given by

$$\| f \|_p = \left(\sum_{u \in V} |f(u)|^p \right)^{1/p}, \quad 1 \leq p < \infty.$$

$$\| f \|_\infty = \max_{u \in V} (|f(u)|), \quad p = \infty.$$