Signal Processing 98 (2014) 88–95

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

Sampling theorems in function spaces for frames associated with linear canonical transform

Jun Shi [a,*], Xiaoping Liu [a,**], Qinyu Zhang [b], Naitong Zhang [a,b]

[a] Communication Research Center, Harbin Institute of Technology, Harbin 150001, China
[b] Shenzhen Graduate School, Harbin Institute of Technology, Shenzhen 518055, China

ARTICLE INFO

Article history:
Received 31 January 2013
Received in revised form
22 August 2013
Accepted 12 November 2013
Available online 23 November 2013

Keywords:
Linear canonical transform
Frames
Riesz bases
Function spaces
Sampling theorem

ABSTRACT

The linear canonical transform (LCT) has proven to be a powerful tool in optics and signal processing. Most existing sampling theories of this transform were derived from the LCT band-limited signal viewpoint. However, in the real world, many analog signals encountered in practical engineering applications are non-bandlimited. The purpose of this paper is to derive sampling theorems of the LCT in function spaces for frames without band-limiting constraints. We extend the notion of shift-invariant spaces to the LCT domain and then derive a sampling theorem of the LCT for regular sampling in function spaces with frames. Further, the theorem is modified to the shift sampling in function spaces by using the Zak transform. Sampling and reconstructing signals associated with the LCT are also discussed in the case of Riesz bases. Moreover, some examples and applications of the derived theory are presented. The validity of the theoretical derivations is demonstrated via simulations.

1. Introduction

The linear canonical transform (LCT) [1] is also known as ABCD transform, generalized Fresnel transform, generalized Huygens integral, affine Fourier transform, and quadratic phase systems. Many signal processing operations, such as the Fourier transform (FT), the fractional Fourier transform (FRFT), the Fresnel transform, and the scaling operations are special cases of this transform. The LCT has proven to be a powerful tool for optical systems, gradient-index medium system analysis, filter design, time–frequency analysis, radar system analysis, pattern recognition, communications, and many others [2–14]. The continuous-time LCT of a signal or function, $f(t) \in L^2(\mathbb{R})$, is defined as [2]

$$F_M(u) = \mathcal{L}^M\{f(t)\}(u) = \begin{cases} \int_{\mathbb{R}} f(t)\mathcal{K}_M(u,t)\,dt, & b \neq 0 \\ \sqrt{d}e^{(jcd/2)u^2} f(du), & b = 0 \end{cases} \quad (1)$$

where $\mathcal{L}^M$ denotes the LCT operator, and kernel $\mathcal{K}_M(u,t)$ is given by

$$\mathcal{K}_M(u,t) = A_b e^{(ja/2b)t^2 + (jd/2b)u^2 - (j/b)ut} \quad (2)$$

where $M = (a,b,c,d)$, $a,b,c,d \in \mathbb{R}$ satisfying $ad - bc = 1$, and $A_b = 1/\sqrt{j2\pi b}$. The u-axis is regarded as the LCT domain. In general, we only consider the case of $b \neq 0$, since the LCT with $b = 0$ is just a chirp multiplication operation. Conversely, the inverse LCT is expressed as $f(t) = \int_{\mathbb{R}} F_M(u)\mathcal{K}_M^*(u,t)\,du$, where $*$ in the superscript denotes the complex conjugate.

In digital signal and image processing, digital communications, etc., a continuous signal is usually represented by its discrete samples. A natural question is how to represent a continuous signal in terms of a discrete sequence. For a band-limited signal, Stern [15] found a Shannon-type sampling theorem associated with the LCT, which provides an exact representation by the signal's uniform samples with sampling

* Corresponding author.
** Corresponding author. Tel.: +86 183 45176 690.
E-mail addresses: dr.junshi@gmail.com (J. Shi), xp.liu@hit.edu.cn (X. Liu), zqy@hit.edu.cn (Q. Zhang), ntzhang@hit.edu.cn (N. Zhang).

0165-1684/$ - see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.sigpro.2013.11.013

rate higher than its Nyquist rate. Although Stern's sampling theory has had an enormous impact [16–22], it has a number of problems: It relies on the use of ideal filters; the bandlimited hypothesis is in contradiction with the idea of a finite duration signal; the bandlimiting operation generates Gibbs oscillations, and finally, the sinc function has a very slow decay rate at infinity such that the computation in the signal domain is very inefficient. Moreover, many applied problems impose different a priori constraints on the type of signals. For these reasons, the sampling and reconstruction problems associated with the LCT have been investigated in function spaces. In [23], Liu et al. established sampling formulae of the LCT for non-bandlimited signals by introducing certain types of non-bandlimited function spaces. Unfortunately, as pointed out by authors of [23], there are no normative rules for determining the parameters of the non-bandlimited function spaces in practical implementations at present.

In our recent paper [24], we proposed a sampling theorem for the LCT in some function space of the form $\mathcal{V}_M(\phi) = \overline{\text{span}}_{L^2}\{\phi(t-n)e^{-(ja/2b)(t^2-n^2)}\}_{n\in\mathbb{Z}}$ without band-limiting constraints, which can provide a suitable and realistic model of sampling and reconstruction for real applications. However, the results were derived for Riesz bases. The contributions of this paper, compared to our previous work, are threefold:

(1) We derive some properties of the function space $\mathcal{V}_M(\phi)$ with frame generators, which extend conventional shift-invariant spaces.
(2) We establish a sampling theorem of the LCT for regular sampling in $\mathcal{V}_M(\phi)$ with frames. Then, the theorem is modified to the shift sampling in $\mathcal{V}_M(\phi)$ by using the Zak transform.
(3) Some necessary conditions for sampling in the LCT domain are established in frame sense and in Riesz basis sense.

The outline of this paper is organized as follows. Section 2 introduces notation and then gives a generalization of the Zak transform in the LCT domain. Section 3 presents some properties of the function spaces related to the LCT and then derives a sampling theorem for the LCT in the function spaces with frames. Further, a shift-sampling theorem for the LCT is established by using the Zak transform. In addition, some relationships and properties about the relevant functions for sampling in the LCT domain are attained in frame sense and in Riesz basis sense. Some examples and applications of the derived results are also discussed. Finally, concluding remarks are given in Section 4.

2. Preliminaries

2.1. Notation

For a measurable function $f(t)$ on $\mathbb{R}$, let

$$\|f(t)\|_\infty = \text{ess sup}|f(t)| \quad \text{and} \quad \|f(t)\|_0 = \text{ess inf}|f(t)| \tag{3}$$

be the essential supremum and infimum of $|f(t)|$, respectively. The characteristic function of a measurable subset $E\subset\mathbb{R}$ is given by

$$\chi_E(t) = \begin{cases} 1, & t\in E \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

2.2. A generalization of the Zak transform associated with the LCT

One of the important tools used in the study of sampling theory is the Zak transform (ZT) [27]. Here, we give a generalization of the ZT associated with the LCT, which will be used in this paper.

The ordinary ZT can be defined in terms of the time-shift operator T_τ as

$$Z_f(\sigma,\omega) = \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}}(\mathsf{T}_{-\sigma}f)(n)e^{-j\omega n} \tag{5}$$

where T_τ is given by $(\mathsf{T}_\tau f)(\cdot) \triangleq f(\cdot-\tau)$. Since the canonical time-shift operator T_τ^M [13] generalizes the ordinary one T_τ, it naturally lends itself to defining a linear canonical ZT (LCZT), i.e.,

$$Z_f^M(\sigma,u) = \sum_{n\in\mathbb{Z}}(\mathsf{T}_{-\sigma}^M f)(n)\mathcal{K}_M(u,n) \tag{6}$$

where $(\mathsf{T}_\tau^M f)(\cdot) \triangleq f(\cdot-\tau)e^{-(ja/b)\tau(\cdot-\tau/2)}$ [13]. Whenever $M=(0,1,-1,0)$, (6) reduces to the ordinary ZT. If $\sigma=0$, (6) is identical with the discrete-time LCT (DTLCT) defined in (6) of [24].

3. Main results

3.1. Sampling in function spaces for frames associated with the LCT

For any function $\phi(t)\in L^2(\mathbb{R})$, let $\{\phi(t-n)\}_{n\in\mathbb{Z}}$ be a set of functions from $L^2(\mathbb{R})$ and $\mathcal{V}(\phi) = \overline{\text{span}}_{L^2}\{\phi(t-n)\}$ the closed subspace of $L^2(\mathbb{R})$ spanned by $\{\phi(t-n)\}_{n\in\mathbb{Z}}$. From [25], the basic principle behind the conventional shift-invariant space $\mathcal{V}(\phi)$ is that if $f(t)\in\mathcal{V}(\phi)$, then $(\mathsf{T}_\tau f)(t)\in\mathcal{V}(\phi)$ for all $\tau\in\mathbb{Z}$. Similarly, the subspace $\mathcal{V}_M(\phi) = \overline{\text{span}}_{L^2}\{\phi(t-n)e^{-(ja/2b)(t^2-n^2)}\}_{n\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ constructed in [24] has a following basic property:

$$f(t)\in\mathcal{V}_M(\phi) \Leftrightarrow (\mathsf{T}_\tau^M f)(t)\in\mathcal{V}_M(\phi) \tag{7}$$

for all $\tau\in\mathbb{Z}$, where T_τ^M is defined in (6). Whenever $M=(0,1,-1,0)$, $\mathcal{V}_M(\phi)$ and T_τ^M reduces to $\mathcal{V}(\phi)$ and T_τ, respectively. Clearly, $\mathcal{V}(\phi)$ is just a special case of $\mathcal{V}_M(\phi)$, and the relationship between them is given by

$$f(t)\in\mathcal{V}_M(\phi) \Leftrightarrow f(t)e^{(ja/2b)t^2}\in\mathcal{V}(\phi). \tag{8}$$

Consequently, $\{\phi_{n,M}(t) \triangleq \phi(t-n)e^{-(ja/2b)(t^2-n^2)}\}_{n\in\mathbb{Z}}$ is a frame or a Riesz basis for $\mathcal{V}_M(\phi)$ if and only if $\{\phi(t-n)\}_{n\in\mathbb{Z}}$ is a frame or a Riesz basis for $\mathcal{V}(\phi)$. For simplicity, let

$$G_{\phi,M}(u) = \sum_{k\in\mathbb{Z}}\left|\Phi\left(\frac{u}{b}+2k\pi\right)\right|^2 \tag{9}$$

where $\Phi(u/b)$ denotes the FT (with its argument scaled by $1/b$) of $\phi(t)$. It is easy to see that $G_{\phi,M}(u) = G_{\phi,M}(u+2\pi b)$, and $G_{\phi,M}(u)\in L^1(I)$ where $I \triangleq [0,2\pi b]$. Also, let $E_{\phi,M} \triangleq$