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# A sampling theorem for the fractional Fourier transform without band-limiting constraints

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ABSTRACT

The fractional Fourier transform (FRFT), a generalization of the Fourier transform, has proven to be a powerful tool in optics and signal processing. Most existing sampling theories of the FRFT consider the class of band-limited signals. However, in the real world, many analog signals encountered in practical engineering applications are non-bandlimited. The purpose of this paper is to propose a sampling theorem for the FRFT, which can provide a suitable and realistic model of sampling and reconstruction for real applications. First, we construct a class of function spaces and derive basic properties of their basis functions. Then, we establish a sampling theorem without band-limiting constraints for the FRFT in the function spaces. The truncation error of sampling is also analyzed. The validity of the theoretical derivations is demonstrated via simulations.



## 1. Introduction

The fractional Fourier transform (FRFT), which generalizes the Fourier transform (FT), has received much attention in recent years due to its numerous applications [1–5], including in the areas of optics, signal and image processing, communications, etc. The FRFT of a continuous signal or function $f(t)$ is defined as [2]

$$F_\alpha(u) = \mathcal{F}^\alpha\{f(t)\}(u) = \int_{\mathbb{R}} f(t)\mathcal{K}_\alpha(u,t)\,dt \tag{1}$$

where $\mathcal{F}^\alpha$ denotes the FRFT operator, and kernel $\mathcal{K}_\alpha(u,t)$ is given by

$$\mathcal{K}_\alpha(u,t) = \begin{cases} A_\alpha e^{(j(u^2+t^2)/2)\cot\alpha - jut\csc\alpha}, & \alpha \neq k\pi \\ \delta(t-u), & \alpha = 2k\pi \\ \delta(t+u), & \alpha = (2k-1)\pi \end{cases} \tag{2}$$

where $A_\alpha = \sqrt{(1-j\cot\alpha)/2\pi}$, $k \in \mathbb{Z}$, $\cot\alpha = \cos\alpha/\sin\alpha$, and $\csc\alpha = 1/\sin\alpha$. For $\alpha \in [-\pi, \pi]$, the square root factor $A_\alpha$ can be rewritten without ambiguity as [1]

$$A_\alpha = \frac{e^{-j\left[\frac{\alpha}{2} - \frac{\pi}{4}\mathrm{sgn}(\alpha)\right]}}{\sqrt{2\pi|\sin\alpha|}} \tag{3}$$

where $\mathrm{sgn}(\cdot)$ denotes the sign function. When $\alpha$ is outside the interval $[-\pi, \pi]$, we simply need to replace $\alpha$ by its modulo $2\pi$ equivalent lying in this interval and use this value in (3). The $u$ axis is regarded as the fractional Fourier domain. The inverse FRFT with respect to angle $\alpha$ is the FRFT with angle $-\alpha$, i.e., $f(t) = \mathcal{F}^{-\alpha}\{F_\alpha(u)\}(t) = \int_{\mathbb{R}} F_\alpha(u) \mathcal{K}^*_\alpha(u,t)\,du$, where $*$ in the superscript denotes the complex conjugate. In general, we only consider the case of $0 < \alpha < \pi$, since the definition can easily be extended outside the interval $[0, \pi]$ by noting that $\mathcal{F}^{2\pi n}$ is the identity operator for any integer $n$ and that the FRFT operator is additive in angle, i.e., $\mathcal{F}^{\alpha_1+\alpha_2} = \mathcal{F}^{\alpha_1}\mathcal{F}^{\alpha_2}$. Whenever $\alpha = \pi/2$, (1) reduces to the FT given by

$$F(\omega) = \mathfrak{F}\{f(t)\}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(t)e^{-j\omega t}\,dt \tag{4}$$

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**Nomenclature**

| | |
|---|---|
| FT | Fourier transform |
| DTFT | discrete-time Fourier transform |
| FRFT | fractional Fourier transform |
| DTFRFT | discrete-time fractional Fourier transform |
| $\mathfrak{F}$ | FT operator |
| $\mathcal{F}^\alpha$ | FRFT operator |
| $\tilde{\mathcal{F}}^\alpha$ | DTFRFT operator |
| $F(u \csc\alpha)$ | FT (with its argument scaled by $\csc\alpha$) of $f(t)$ |
| $\tilde{F}(u \csc\alpha)$ | DTFT (with its argument scaled by $\csc\alpha$) of $f[n]$ |
| $F_\alpha(u)$ | FRFT of $f(t)$ |
| $\tilde{F}_\alpha(u)$ | DTFRFT of $f[n]$ |
| $\Theta_\alpha$ | continuous fractional convolution operator |
| $\overset{s}{\Theta}_\alpha$ | semi-discrete fractional convolution operator |

with $f(t) \in L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$, where $\mathfrak{F}$ indicates the FT operator. Conversely, the inverse FT is written as $f(t) = (1/\sqrt{2\pi}) \int_{\mathbb{R}} F(\omega) e^{j\omega t}\, d\omega$. It follows that the FRFT exists, for $\alpha$ not multiple of $\pi$, whenever the FT of $f(t)e^{(j/2)t^2 \cot\alpha}$ exists. Since the complex exponent in (2) has a constant magnitude, the FRFT can also be defined in most domains in which the FT can be defined.

In digital signal and image processing, digital communications, etc., a continuous signal is usually represented by its discrete samples. Then, a fundamental problem of FRFT theory is how to represent a continuous signal in terms of a discrete sequence. For a fractional band-limited signal $f(t)$, Xia [6] found a Shannon-type sampling theorem for the FRFT. In particular, the sampling process of this theorem for a $\pi \sin\alpha$-fractional bandlimited signal can be viewed as an approximation procedure in the space of fractional band-limited functions:

$$\mathcal{B}_\alpha = \left\{ \sum_{n\in\mathbb{Z}} f[n]\mathrm{sinc}(t-n)e^{-j((t^2-n^2)/2)\cot\alpha} | f[n] \in \ell^2(\mathbb{Z}) \right\} \tag{5}$$

where $\mathrm{sinc}(\cdot) \triangleq \sin \pi(\cdot)/\pi(\cdot)$. Xia's sampling theorem provides an exact representation by the signal's uniform samples $\{f[n]\}_{n\in\mathbb{Z}}$ and has been currently generalized to many other forms. Zayed and García derived a new sampling expansion using the Hilbert transform in [7]. In [8], Stern extended Xia's result to the generalized form of the FRFT, which is called the linear canonical transform [1]. Tao et al. discussed sampling and sampling rate conversion of band-limited signals in the fractional Fourier domain in [9]. Bhandari and Marziliano [10] proposed a uniform sampling and reconstruction algorithm for sparse signals in the fractional Fourier domain. Furthermore, authors in [11,12] studied multi-channel sampling for the FRFT. However, these extensions and modifications of Xia's sampling theorem [6] were derived from the band-limited signal viewpoint. In the real world, many analog signals encountered in practical engineering applications are non-bandlimited. Recently, Liu et al. [13] introduced new sampling formulae of the generalized FRFT for non-bandlimited signals by constructing a class of function spaces $B^{h,m}_{M,\Omega_M}$ $(m = 1, 2, 3)$. Unfortunately, as the authors of [13] pointed out, there are no normative rules at present for determining the parameters $M$, $h$, $\Omega_M$ in practical implementations.

The purpose of this paper is to propose a sampling theorem associated with the FRFT, which can provide a suitable and realistic model of sampling and reconstruction for real applications. First, we introduce a class of function spaces with a single generator and derive basic properties of their basis functions. Then, we derive a sampling theorem for the FRFT in the function spaces. Moreover, the truncation error of sampling and some potential applications of the derived results are presented. The validity of the theoretical derivations is demonstrated via simulations.

The outline of this paper is organized as follows. In Section 2, notations and some facts for the FRFT are first introduced, and then the concept of fractional convolution is given. In Section 3, a sampling theorem for the FRFT without band-limiting constraints is established, and the truncation error of sampling and some potential applications are also discussed. Finally, concluding remarks are given in Section 4.

## 2. Preliminaries

### 2.1. Notation

Throughout this paper, we consider real-valued signals. Continuous signals are denoted with parentheses, e.g., $f(t)$, $t \in \mathbb{R}$, and discrete signals with brackets, e.g., $c[n]$, $n \in \mathbb{Z}$. We denote the inner $L^2$-product between $f(t)$ and $g(t)$ by

$$\langle f, g\rangle_{L^2} = \int_{\mathbb{R}} f(t) g^*(t)\, dt, \tag{6}$$

and the $\ell^2$-inner product between two sequences $c[n]$ and $d[n]$ by

$$\langle c, d\rangle_{\ell^2} = \sum_{n\in\mathbb{Z}} c[n] d^*[n]. \tag{7}$$

Correspondingly, we denote the $L^2$-norm by $\|f\|^2_{L^2} = \langle f, f\rangle_{L^2}$, and the $\ell^2$-norm by $\|c\|^2_{\ell^2} = \langle c, c\rangle_{\ell^2}$.

Let $\mathcal{H}$ be a Hilbert space and $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$ be a complete set of functions in $\mathcal{H}$. The set is a Riesz basis for $\mathcal{H}$ if and only if there exist constants $0 < A \le B < +\infty$ such that [14]

$$A\|c[n]\|^2_{\ell^2} \le \left\| \sum_{n\in\mathbb{Z}} c[n]\varphi_n(t) \right\|^2_{L^2} \le B\|c[n]\|^2_{\ell^2}, \quad \forall c[n] \in \ell^2(\mathbb{Z}) \tag{8}$$

with equality if and only if the basis is orthonormal, i.e., when $A = B = 1$.

For a measurable function $f(t)$ on $\mathbb{R}$, let $\|f(t)\|_\infty = \mathrm{ess\,sup}|f(t)|$ and $\|f(t)\|_0 = \mathrm{ess\,inf}|f(t)|$ be the essential supremum and infimum of $|f(t)|$, respectively.