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Fast communication

On the perturbation of measurement matrix in non-convex compressed sensing

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ABSTRACT

We study l_p $(0<p<1)$ minimization under both additive and multiplicative noise. Theorems are presented for completely perturbed l_p $(0<p<1)$ minimization. Theorems reveal that under suitable conditions the stability of l_p minimization with certain values of $0<p<1$ is limited by the noise level in the observation. The restricted isometry property condition and the worst case reconstruction error bound are given in terms of restricted isometry constant and relative perturbations. Simulation results are presented and compared to state-of-the-art methods.

1. Introduction

Compressed sensing (CS) aims to recover a sparse or near-sparse signal $x \in R^n$ from $m<n$ linear measurements

$$y = \Phi x$$

where $\Phi \in R^{m \times n}$ is the measurement matrix modeling the measurement system. In practice measurements y are corrupted with additive noise e, therefore a perturbed measurement vector in the form of

$$\hat{y} = \Phi x + e \tag{1}$$

is considered.

However in standard CS, the measurement matrix Φ is assumed known *a priori*. Consider that the measurement matrix Φ is corrupted by a perturbation E. The replacement of Φ by $\Phi+E$ in (1) introduces a multiplicative noise term Ex in the measurements in addition to additive noise. This situation can be encountered in several applications. For example, the quantization operation during the implementation of the measurement matrix in a sensor causes matrix perturbation. Furthermore, in radar imaging [1] and communication problems [2] when Φ represents a system model, E can model the system perturbation. Also hardware imperfections due to the non-exact component values used in the low-pass filter section of a random demodulator can be another reason for matrix perturbation [3,4].

In [5], stability analysis of the basis pursuit denoising (BPDN) is given for the completely perturbed CS problem. Here, completely perturbed means that measurements are corrupted with additive noise as well as multiplicative noise. This completely perturbed framework is based on the relative error bounds of the measurement matrix and additive noise. It is shown that the signal recovery is robust and the recovery error is linearly proportional to the perturbation level. Similar recovery results are also presented in [6–8].

In this paper a completely perturbed CS scenario is considered for l_p $(0<p<1)$ minimization. The main result

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is that l_p $(0<p<1)$ minimization for the completely perturbed scenario is stable under suitable conditions.

1.1. Notations and symbols

This paper uses the similar notations and techniques of [5]. Let the perturbations E and e be quantified with the relative bounds

$$\frac{\|E\|_2}{\|\Phi\|_2} \le \varepsilon_\Phi, \quad \frac{\|E\|_2^{(k)}}{\|\Phi\|_2^{(k)}} \le \varepsilon_\Phi^{(k)}, \quad \frac{\|e\|_2}{\|y\|_2} \le \varepsilon_y, \tag{2}$$

$\|\cdot\|_2$ denotes the spectral norm and $\|\cdot\|_2^{(k)}$ represents the largest spectral norm taken over all k-column sub-matrices. r_k and s_k in (3) define the signal's tail relative to its head. x_k is the best k-term approximation to x and $x_{k^c} = x - x_k$.

$$r_k := \frac{\|x_{k^c}\|_2}{\|x_k\|_2}, \quad s_k := \frac{\|x_{k^c}\|_1}{\sqrt{k}\|x_k\|_2} \tag{3}$$

1.2. CS background

In standard CS recovery, the reconstructed signal $\hat{x}$ is the solution to the basis pursuit denoising (BPDN) problem

$$\min \|\hat{x}\|_1 \quad \text{subject to } \|\hat{y} - \Phi\hat{x}\|_2 \le \varepsilon,$$

where ε is the upper bound level of the noise term e in (1), and can be solved by using convex optimization techniques such as interior-point methods or homotopy methods. Candès and Tao [9] show that the stable recovery of BPDN is based on a special matrix property called restricted isometry property (RIP).

Definition 1. A matrix Φ satisfies the RIP of order k if there exists a constant $\delta_k \in (0,1)$ such that

$$(1-\delta_k)\|x\|_2^2 \le \|\Phi x\|_2^2 \le (1+\delta_k)\|x\|_2^2$$

holds for all k-sparse signals x with restricted isometry constant (RIC) δ_k.

Theorem 1 (*Candès [10]*). *Assume that $\delta_{2k} < \sqrt{2}-1$. Then the reconstructed signal $\hat{x}$ to the BPDN problem satisfies*

$$\|x-\hat{x}\|_2 \le C_0\varepsilon + C_1 k^{-1/2}\|x - x_k\|_1$$

where

$$C_0 = \frac{4\sqrt{1+\delta_{2k}}}{1-(\sqrt{2}+1)\delta_{2k}} \quad \text{and} \quad C_1 = \frac{2[1+(\sqrt{2}-1)\delta_{2k}]}{1-(\sqrt{2}+1)\delta_{2k}}.$$

Theorem 1 states that it is possible to recover a k-sparse signal $(x = x_k)$ provided that the measurement matrix Φ satisfies $\delta_{2k} < \sqrt{2}-1$.

1.3. Completely perturbed BPDN

In the completely perturbed CS, the following theorem for BPDN is stated in [5].

Theorem 2. *Let the relative perturbations ε_Φ, $\varepsilon_\Phi^{(k)}$, $\varepsilon_\Phi^{(2k)}$ and ε_y be as in (2). Define the constants $\kappa_\Phi^{(k)} = \sqrt{1+\delta_k}/\sqrt{1-\delta_k}$ and $\gamma_\Phi = \|\Phi\|_2/\sqrt{1-\delta_k}$ due to matrix Φ.*

Suppose that the signal x satisfies $r_k + s_k < 1/\kappa_\Phi^{(k)}$. If RIC for Φ satisfies

$$\delta_{2k} < \frac{\sqrt{2}}{(1+\varepsilon_\Phi^{(2k)})^2} - 1 \tag{4}$$

then the solution $\hat{z}$ to

$$\min \|\hat{z}\|_1 \quad \text{subject to } \|\hat{y} - A\hat{z}\|_2 \le \varepsilon'_{\Phi,k,y}$$

using $A = \Phi + E$ satisfies

$$\|\hat{z} - x\|_2 \le \hat{C}_0\varepsilon'_{\Phi,k,y} + \hat{D}_0\frac{\|x-x_k\|_1}{k^{1/2}}$$

with total noise parameter

$$\varepsilon'_{\Phi,k,y} = \left(\frac{\varepsilon_\Phi^{(k)}\kappa_\Phi^{(k)} + \varepsilon_\Phi\gamma_\Phi r_k}{1-\kappa_\Phi^{(k)}(r_k+s_k)} + \varepsilon_y\right)\|y\|_2$$

for some constants $\hat{C}_0$ and $\hat{D}_0$.

Theorem 2 denotes that if matrix perturbation E is small then the signal recovery is robust and the recovery error grows linearly with the perturbation level. Note that $\varepsilon_\Phi^{(2k)} < \sqrt[4]{2}-1$ since $\delta_{2k} \ge 0$.

2. Completely perturbed l_p $(0<p<1)$ minimization

Recently, there has been great interest in sparse recovery problem in CS using non-convex minimization methods. Focal undetermined system solver (FOCUSS) [11] is proposed in the solution of BP by replacing the objective l_1 norm with l_p norm. It is solved using iteratively reweighted least squares (IRLS). In [12] regularized IRLS is used in the solution and it is shown that l_p $(0<p<1)$ minimization reconstructs sparse signal exactly with fewer measurements compared to unregularized IRLS. Saab and Yılmaz [13] studied the stability and robustness of l_p $(0<p<1)$ minimization and it is shown that the sufficient conditions for exact reconstruction are weaker. Their results also indicate that the exact reconstruction is possible with fewer measurements compared to BP.

In this section theoretical results for l_p $(0<p<1)$ minimization are presented for completely perturbed CS. Sufficient conditions for stable recovery of l_p $(0<p<1)$ minimization are given. Before proving our main results, we utilize a lemma from [5].

Lemma 1. *Assume that RIC for matrix Φ is δ_k and relative perturbation $\varepsilon_\Phi^{(k)}$ is associated with matrix E. Then the RIC $\hat{\delta}_k$ for matrix A satisfies*

$$\hat{\delta}_k \le (1+\delta_k)(1+\varepsilon_\Phi^{(k)})^2 - 1$$

Theorem 3. *Let the relative perturbations ε_Φ, $\varepsilon_\Phi^{(2ak)}$, and ε_y be as in (2) and suppose that the signal x satisfies $r_k + s_k < 1/\kappa_\Phi^{(k)}$. If RIC for Φ satisfies*

$$\delta_{2ak} < \frac{2+\sqrt{2}a^{1/2-1/p}}{(1+\sqrt{2}a^{1/2-1/p})(1+\varepsilon_\Phi^{(2ak)})^2} - 1 \tag{5}$$